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Factorized finite-size Ising model spin matrix elements from separation of variables

G von Gehlen[1], N Iorgov[2], S Pakuliak[3,4] and V Shadura[2]

[1] Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Germany
[2] Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine
[3] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia
[4] Institute of Theoretical and Experimental Physics, Moscow 117259, Russia

E-mail: gehlen@th.physik.uni-bonn.de, iorgov@bitp.kiev.ua, pakuliak@theor.jinr.ru and shadura@bitp.kiev.ua

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Abstract
Using the Sklyanin–Kharchev–Lebedev method of separation of variables adapted to the cyclic Baxter–Bazhanov–Stroganov or the $\tau^{(2)}$-model, we derive factorized formulae for general finite-size Ising model spin matrix elements, proving a recent conjecture by Bugrij and Lisovyy.

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1. Introduction

Much work has been done on the two-dimensional Ising model (IM) during the past 60 years. Many analytic results for the partition function and correlations have been obtained. These have greatly contributed to establish our present understanding of continuous phase transitions in systems with short-range interactions [1–6]. Recent overviews with many references are given, e.g. in [7–9]. Many rather different mathematical approaches have been used, so that already 30 years ago Baxter and Enting published the '399th' solution for the free energy [10] (see also [11]). Spin–spin correlation functions can be written as Pfaffians of Toeplitz determinants. Most work has focused on the thermodynamic limit and scaling properties since these give contact to field theoretical results and to beautiful Painlevé properties [5, 6, 12].

Only during the last decade has more attention been drawn to correlations and spin matrix elements (form factors [13]) in finite-size Ising systems [14–16]. Nanophysics experimental arrangements often deal with systems where the finite size matters. Recent theoretical work on the finite-size IM started from Pfaffians and related Clifford approaches. In [17] it has been pointed out that one may write completely factorized closed expressions for spin matrix elements of finite-size Ising systems. One goal of the present paper is to prove the beautiful compact formula conjectured in equation (12) of [17], see (129). For achieving this, we introduce a method which has not yet been applied to the Ising model: separation of variables (SoV) for cyclic quantum spin systems. Our approach is the adaption to cyclic models of the method introduced by Sklyanin [19, 20] and further developed by Kharchev and Lebedev [21, 22]. We also make extensive use of the analysis of quantum cyclic systems given in [23].

Little is known about state vectors of the two-dimensional finite-size IM. Only partial information about these state vectors can be obtained from the work of [3]. Recently Lisovyy [24] found explicit expressions using the Grassmann algebra method. Here we shall present our SoV approach [25–27] which gives explicit formulae for finite-size state vectors too. However, these come in a basis quite different from the one used in [24]. We shall calculate spin matrix elements by directly sandwiching the spin operator between state vectors. Factorized expressions result if we manage to perform the multiple spin summations over the intermediate states.

The prototype of a general N-state cyclic spin model is the Baxter–Bazhanov–Stroganov model (BBS) [28–30], also known as the $\tau^{(2)}$-model. The standard IM is a very special degenerate case of the BBS model. In order to avoid formulating many precautions necessary when dealing with the very special IM, we shall develop our version of the SoV machinery considering the general BBS model. We chose to do this also because of the great interest in the BBS model due to the fact that its transfer matrix commutes with the integrable Chiral Potts model (CPM) [31, 32] transfer matrix [29, 33]. Obtaining state vectors for the CPM is a great actual challenge [34, 35]. Although the eigenvectors for the transfer matrix of the BBS model with periodic boundary condition are unknown for $N > 2$, explicit formulae for the eigenvectors of the BBS model with open and fixed boundary conditions have been found [36, 37].

This paper is organized as follows: in section 2 we define the BBS model and its Ising specializations. In section 3 we discuss the Sklyanin SoV method adapted to the BBS model as a cyclic system. We start with the necessary first step, the solution of the associated auxiliary problem. In a second step we obtain the eigenvectors and eigenvalues of the periodic system by Baxter equations. The conditions which ensure that the Baxter equations have non-trivial solutions are formulated as truncated functional equations. Section 4 gives a description of local spin operators in terms of global elements of the monodromy matrix. Starting with section 5 we restrict ourselves to the case $N = 2$, for which the BBS model becomes a generalized five-parameter plaquette Ising model. In section 6 we further specialize to the homogeneous case and then to the two-parameter Ising case. Periodic boundary condition eigenvectors are explicitly constructed. Section 7 is devoted to our main result, the proof of the factorized formula for Ising spin matrix elements between arbitrary finite-size states. This is shown to agree with the Bugrij–Lisovyy conjecture. In section 8 we give an analogous formula for the Ising quantum chain in a transverse field. Finally, section 9 presents our conclusions. A large part of this paper relies on our work in [25–27]. Sections 4, 2.2 and 6.2 give new material.

2. The BBS $\tau^{(2)}$ model

2.1. The inhomogeneous BBS model for general N

We define the BBS model as a quantum chain model. To each site k of the quantum chain we associate a cyclic L-operator [29, 30] acting in a two-dimensional auxiliary space

$$L_k(\lambda) = \begin{pmatrix} 1 + \lambda\varkappa_k \mathbf{v}_k, & \lambda \mathbf{u}_k^{-1}(a_k - b_k\mathbf{v}_k) \\ \mathbf{u}_k(c_k - d_k\mathbf{v}_k), & \lambda a_k c_k + \mathbf{v}_k b_k d_k/\varkappa_k \end{pmatrix}, \qquad k = 1, 2, \ldots, n. \quad (1)$$

λ is the spectral parameter, n is the number of sites. There are five parameters $\varkappa_k, a_k, b_k, c_k, d_k$ per site. $\mathbf{u}_k$ and $\mathbf{v}_k$ are elements of an ultra local Weyl algebra, obeying

$$\mathbf{u}_j\mathbf{u}_k = \mathbf{u}_k\mathbf{u}_j, \qquad \mathbf{v}_j\mathbf{v}_k = \mathbf{v}_k\mathbf{v}_j, \qquad \mathbf{u}_j\mathbf{v}_k = \omega^{\delta_{j,k}}\mathbf{v}_k\mathbf{u}_j, \qquad \omega = \mathrm{e}^{2\pi\mathrm{i}/N}, \qquad \mathbf{u}_k^N = \mathbf{v}_k^N = 1.$$

At each site k we define an N-dimensional linear space (quantum space) $\mathcal{V}_k$ with the basis $|\gamma\rangle_k, \gamma \in \mathbb{Z}_N$, the dual space $\mathcal{V}_k^*$ with the basis ${}_k\langle\gamma|, \gamma \in \mathbb{Z}_N$, and the natural pairing ${}_k\langle\gamma'|\gamma\rangle_k = \delta_{\gamma',\gamma}$. In $\mathcal{V}_k$ and $\mathcal{V}_k^*$ the Weyl elements $\mathbf{u}_k$ and $\mathbf{v}_k$ act by the formulae:

$$\mathbf{u}_k|\gamma\rangle_k = \omega^\gamma|\gamma\rangle_k, \qquad \mathbf{v}_k|\gamma\rangle_k = |\gamma+1\rangle_k; \qquad {}_k\langle\gamma|\mathbf{u}_k = {}_k\langle\gamma|\omega^\gamma, \qquad {}_k\langle\gamma|\mathbf{v}_k = {}_k\langle\gamma-1|. \tag{2}$$

The monodromy $T_n(\lambda)$ and transfer matrix $\mathbf{t}_n(\lambda)$ for the n sites chain are defined as

$$T_n(\lambda) = L_1(\lambda)\cdots L_n(\lambda) = \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ C_n(\lambda) & D_n(\lambda) \end{pmatrix}, \qquad \mathbf{t}_n(\lambda) = \operatorname{tr} T_n(\lambda) = A_n(\lambda) + D_n(\lambda). \tag{3}$$

This quantum chain is integrable since the L-operators (1) are intertwined by the twisted 6-vertex R-matrix at root of unity

$$R(\lambda,\nu) = \begin{pmatrix} \lambda-\omega\nu & 0 & 0 & 0 \\ 0 & \omega(\lambda-\nu) & \lambda(1-\omega) & 0 \\ 0 & \nu(1-\omega) & \lambda-\nu & 0 \\ 0 & 0 & 0 & \lambda-\omega\nu \end{pmatrix}, \tag{4}$$

$$R(\lambda,\nu)L_k^{(1)}(\lambda)L_k^{(2)}(\nu) = L_k^{(2)}(\nu)L_k^{(1)}(\lambda)R(\lambda,\nu), \tag{5}$$

where $L_k^{(1)}(\lambda) = L_k(\lambda)\otimes\mathbb{I}$, $L_k^{(2)}(\lambda) = \mathbb{I}\otimes L_k(\lambda)$. Relation (5) leads to $[\mathbf{t}_n(\lambda), \mathbf{t}_n(\mu)] = 0$. So $\mathbf{t}_n(\lambda)$ is the generating function for the commuting set of non-local and non-Hermitian Hamiltonians $\mathbf{H}_0, \ldots, \mathbf{H}_n$:

$$\mathbf{t}_n(\lambda) = \mathbf{H}_0 + \mathbf{H}_1\lambda + \cdots + \mathbf{H}_{n-1}\lambda^{n-1} + \mathbf{H}_n\lambda^n. \tag{6}$$

From (5) it also follows that the upper-right entry $B_n(\lambda)$ of $T_n(\lambda)$ is the generating function for another commuting set of operators $\mathbf{h}_1, \ldots, \mathbf{h}_n$:

$$[B_n(\lambda), B_n(\mu)] = 0, \qquad B_n(\lambda) = \mathbf{h}_1\lambda + \mathbf{h}_2\lambda^2 + \cdots + \mathbf{h}_n\lambda^n. \tag{7}$$

Observe that $\mathbf{H}_0$ and $\mathbf{H}_n$ can easily be written explicitly in terms of the global $\mathbb{Z}_N$-charge rotation operator $\mathbf{V}_n$

$$\mathbf{H}_0 = 1 + \mathbf{V}_n\prod_{k=1}^n \frac{b_k d_k}{\varkappa_k}, \qquad \mathbf{H}_n = \prod_{k=1}^n a_k c_k + \mathbf{V}_n\prod_{k=1}^n \varkappa_k, \qquad \mathbf{V}_n = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n. \tag{8}$$

Here we shall not explain the great interest in the BBS model due to a second intertwining relation in the Weyl-space indices found in [29] and the related fact that for particular parameters the Baxter Q-operator of the BBS model is the transfer matrix of the integrable Chiral Potts model, see [29, 33, 38]. We will also not discuss the generalizations of the BBS model introduced by Baxter in [39], and not explain how (1) arises in cyclic representations of the quantum group $U_q(sl_2)$ (see e.g. [23, 40, 41]).

The transfer matrix (3) can be written equivalently as a product over face Boltzmann weights [28, 33]:

$\mathbf{t}_n(\lambda) = \prod_{k=2}^{n+1} W_\tau(\gamma'_{k-1}, \gamma'_k, \gamma_k, \gamma_{k-1})$ where each square plaquette of the lattice (see figure 1) contributes the Boltzmann weights

$$W_\tau(\gamma_{k-1},\gamma_k,\gamma'_{k-1},\gamma'_k) = \sum_{m_{k-1}=0}^{1} \omega^{m_{k-1}(\gamma'_k-\gamma_{k-1})} \times (-\omega t_q)^{\gamma_k-\gamma'_k-m_{k-1}} F'_{k-1}(\gamma_{k-1}-\gamma'_{k-1}, m_{k-1}) F''_k(\gamma_k-\gamma'_k, m_{k-1}) \tag{9}$$

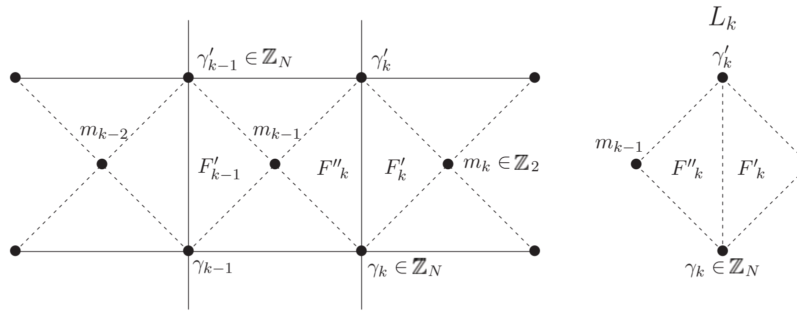

Figure 1. Illustration of the two versions: (left) we see the W of (9) indicated by full lines, whereas the L_k of (1) and (3) arise if we look at the lattice formed by the dashed lines in the left figure and the dashed rhombus shown in the right. The lattice is built by $\mathbb{Z}_N$-spins on the full lines and $\mathbb{Z}_2$-spins in the centers.

where $m_k \in \{0, 1\}$ and $F'_k(\Delta\gamma, m_k) = F''_k(\Delta\gamma, m_k) = 0$ if $\Delta\gamma \neq \{0, 1\}$, and the non-vanishing values are

$$F'_k = \begin{pmatrix} 1 & \lambda a_k \\ \varkappa_k & -b_k/\omega \end{pmatrix}, \qquad F''_k = \begin{pmatrix} 1 & \lambda c_k \\ 1 & -d_k/\varkappa_k \end{pmatrix}. \tag{10}$$

The vanishing of $F'_k(\Delta\gamma, m_k)$ and $F''_k(\Delta\gamma, m_k)$ for $\Delta\gamma \neq \{0, 1\}$ means that the vertically neighboring $\mathbb{Z}_N$-spins cannot differ by more than 1. The equivalence to the transfer matrix defined by (1) and (3) is seen writing the matrix elements of (1) as

$$\langle \gamma'_k | L_k(\lambda)_{m_{k-1}, m_k} | \gamma_k \rangle = \omega^{m_{k-1}\gamma'_k - m_k\gamma_k} \lambda^{\gamma'_k - \gamma_k - m_{k-1}} F''_k(\gamma'_k - \gamma_k, m_{k-1}) F'_k(\gamma'_k - \gamma_k, m_k). \tag{11}$$

2.2. *Homogeneous BBS model for $N = 2$*

The integrability of the BBS model is also valid if the parameters $\varkappa_k, a_k, \ldots, d_k$ vary from site to site and the construction of eigenvalues and eigenvectors can be performed for this general case. However, in order to obtain compact explicit formulae for matrix elements, we shall often make all parameters equal: $\varkappa_k = \varkappa, \ldots, d_k = d$ and call this the homogeneous model. In [42] it has been shown that for $N = 2$ the general homogeneous BBS model can be rewritten as a generalized plaquette Ising model with Boltzmann weights

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = a_0 \left(1 + \sum_{1 \leqslant i < j \leqslant 4} a_{ij}\sigma_i\sigma_j + a_4\sigma_1\sigma_2\sigma_3\sigma_4 \right), \tag{12}$$

subject to the free-fermion condition $a_4 = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

For $N = 2$ the Weyl elements can be represented by Pauli matrices. Fixing $\varkappa = 1$ the L-operator becomes

$$L_k(\lambda) = \begin{pmatrix} 1 + \lambda\sigma_k^x & \lambda\sigma_k^z\left(a - b\sigma_k^x\right) \\ \sigma_k^z\left(c - d\sigma_k^x\right) & \lambda ac + \sigma_k^x bd \end{pmatrix}$$

degenerating at $\lambda = b/a$:

$$L_k(b/a) = \begin{pmatrix} 1 + b\big/a\sigma_k^x \\ \sigma_k^z\left(c - d\sigma_k^x\right) \end{pmatrix} \left(1, \quad b\sigma_k^z\right).$$

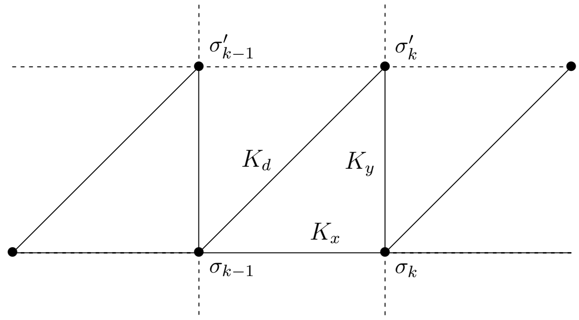

Figure 2. Transfer matrix for the triangular Ising lattice. Solid lines show the interaction between spins.

The matrix elements of the corresponding transfer matrix are

$$\langle\{\sigma'\}|\mathbf{t}_n(b/a)|\{\sigma\}\rangle = \prod_{k=1}^{n}\left(\delta_{\sigma_k,\sigma'_k}(1+bc\sigma_{k-1}\sigma'_k)+\delta_{\sigma_k,-\sigma'_k}b/a(1-ad\sigma_{k-1}\sigma'_k)\right),$$

where $\{\sigma\} = \{\sigma_1,\ldots,\sigma_n\}$ and $\{\sigma'\} = \{\sigma'_1,\ldots,\sigma'_n\}$ are the values of the spin variables of two neighboring rows, $\sigma_k = (-1)^{\gamma_k}$, $\sigma'_k = (-1)^{\gamma'_k} \in \{+1,-1\}$, and the identifications $\sigma_{n+k} = \sigma_k$, $\sigma'_{n+k} = \sigma'_k$ are used.

The matrix elements of the transfer matrix of the Ising model on the triangular lattice (see figure 2) are

$$\langle\{\sigma'\}|\mathbf{t}_\triangle|\{\sigma\}\rangle = \prod_{k=1}^{n}\exp(K_x\sigma_{k-1}\sigma_k + K_y\sigma_k\sigma'_k + K_d\sigma_{k-1}\sigma'_k). \tag{13}$$

The kth factor of this product, taken at $\sigma_k = \sigma'_k$ is

$$\exp(K_y)\exp((K_x+K_d)\sigma_{k-1}\sigma'_k) = \exp(K_y)\cosh(K_x+K_d)(1+\tanh(K_x+K_d)\sigma_{k-1}\sigma'_k),$$

and at $\sigma_k = -\sigma'_k$ is

$$\exp(-K_y)\exp((K_d-K_x)\sigma_{k-1}\sigma'_k) = \exp(-K_y)\cosh(K_d-K_x)(1+\tanh(K_d-K_x)\sigma_{k-1}\sigma'_k).$$

Now it is easy to compare the transfer matrices $\mathbf{t}_n(b/a)$ and $\mathbf{t}_\triangle$:

$$\mathbf{t}_\triangle = \exp(nK_y)\cosh^n(K_x+K_d)\mathbf{t}_n(b/a), \qquad \exp(-2K_y)\frac{\cosh(K_d-K_x)}{\cosh(K_d+K_x)} = b/a,$$

$$\tanh(K_x+K_d) = bc, \qquad \tanh(K_x-K_d) = ad.$$

Although we considered $\mathbf{t}_n(\lambda)$ at the special value of the spectral parameter $\lambda = b/a$, the transfer-matrix eigenstates are independent of this choice of λ. So the eigenstates of the transfer matrix of the Ising model on the triangular lattice appear as eigenstates of the general homogeneous BBS model for $N = 2$ (the parameter $\varkappa$ and one of the parameters $a,\ldots,d$ in the case of homogeneous periodic BBS model can be absorbed by a rescaling of the other parameters and using a diagonal similarity transformation of the L-operators). The formulae for them will be given later. Unfortunately, factorized formulae for the matrix elements of the spin operator in this general case have not been found. There are only two special cases for which such formulas are available:

- The row-to-row transfer-matrix for the Ising model on the square lattice:

$$a = c, \quad b = d: \quad K_d = 0, \quad \mathrm{e}^{-2K_y} = b/a, \quad \tanh K_x = ab. \tag{14}$$

This case will be the main object of our attention. It is the most general case where we have the factorized formula for the spin operator matrix elements found by Bugrij and Lisovyy.

- Diagonal-to-diagonal transfer matrix for the Ising model on the square lattice:

$$a = c, \quad b = -d: \quad K_x = 0, \quad \mathrm{e}^{-2K_y} = b/a, \quad \tanh K_d = ab. \tag{15}$$

It is known [43] that such transfer matrices with different parameters $K_y = L$, $K_d = K$ (and corresponding a, b) constitute a commuting set of matrices having common eigenvectors, provided

$$\sinh 2K \sinh 2L = \frac{a^2 - b^2}{1 - a^2 b^2} = \frac{1}{\mathsf{k}'} \tag{16}$$

is fixed. Thus in this case the eigenvectors depend on k' only. Therefore, in order to find the eigenvectors and the corresponding matrix elements of the spin operator it is sufficient to fix $a = c = 1/(\mathsf{k}')^{1/2}$ and $b = d = 0$ and so to obtain a special case of the formulae for the row-to-row transfer matrix of the Ising model on the square lattice. Note that we get [27] the same matrix elements in the case of the quantum Ising chain in a transverse field with strength k' because the corresponding Hamiltonian commutes with the transfer matrices having the same k'. Another remark: with the restriction $a = c$, $b = -d$, $\varkappa = 1$, the transfer matrices commute among themselves at independent values of *two* spectral parameters: λ and the parameter which uniformizes (16) (a parameter on the elliptic curve with modulus k', see [43]).

3. Separation of variables for the cyclic BBS-model

3.1. Solving the auxiliary system (7): Eigenvalues and eigenvectors of $B_n(\lambda)$

We start giving a summary of the SoV method as applied to the general inhomogeneous $\mathbb{Z}_N$-BBS model [25]. The aim is to find the eigenvalues and eigenstates of the n-site periodic transfer matrix $\mathbf{t}_n(\lambda)$ of (3), and the idea [19–22] is first to construct a basis of the N^n-dimensional eigenspace from eigenstates of $B_n(\lambda)$, see (7). This can be done by a recurrent procedure. Then the eigenstates of $\mathbf{t}_n(\lambda)$ are written as linear combinations of the $B_n(\lambda)$-eigenstates. The multi-variable coefficients are determined by Baxter $T-Q$-equations which by SoV separate into a set of single-variable equations.

From (7) the eigenvalues of $B_n(\lambda)$ are polynomials in the spectral variable λ. Factorizing this polynomial, for $n \geqslant 2$ we obtain

$$B_n(\lambda)|\Psi_{\boldsymbol{\lambda}}\rangle = \lambda\lambda_0 \prod_{k=1}^{n-1} (\lambda - \lambda_k)|\Psi_{\boldsymbol{\lambda}}\rangle; \qquad \boldsymbol{\lambda} = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}, \tag{17}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the $n-1$ zeros of the eigenvalue polynomial and λ_0 is a normalizing factor. We can label the eigenvectors by $\boldsymbol{\lambda}$.

An overview of the space of eigenstates of $B_n(\lambda)$ is easily obtained using the intertwining relations (5). It follows from (5) that the operators $A_n(\lambda)$ and $D_n(\lambda)$ of the monodromy (3), taken at a zero $\lambda = \lambda_k$, are cyclic ladder operators with respect to the kth component of $\boldsymbol{\lambda}$ in $|\Psi_{\boldsymbol{\lambda}}\rangle$. To see this consider e.g. the intertwining relation

$$(\lambda - \omega\mu)A_n(\lambda)B_n(\mu) = \omega(\lambda - \mu)B_n(\mu)A_n(\lambda) + \mu(1-\omega)A_n(\mu)B_n(\lambda), \tag{18}$$

which is a component of (5). Fixing $\lambda = \lambda_k$, $k = 1, \ldots, n-1$, in (18) and acting on $\Psi_{\boldsymbol\lambda}$, the last term in (18) vanishes and we obtain

$$B_n(\mu)\,(A_n(\lambda_k)|\Psi_{\boldsymbol\lambda}\rangle) = \mu\lambda_0(\mu - \omega^{-1}\lambda_k)\prod_{s\neq k}(\mu - \lambda_s)\,(A_n(\lambda_k)|\Psi_{\boldsymbol\lambda}\rangle). \tag{19}$$

This means that

$$A_n(\lambda_k)|\Psi_{\boldsymbol\lambda}\rangle = \varphi_k \cdot \left|\Psi_{\lambda_0,\ldots,\omega^{-1}\lambda_k,\ldots,\lambda_{n-1}}\right\rangle. \tag{20}$$

Later we shall give an explicit expression for the proportionality factor φ_k. Similarly, from another component of (5) and with another factor $\widetilde{\varphi}_k$ we obtain

$$D_n(\lambda_k)|\Psi_{\boldsymbol\lambda}\rangle = \widetilde{\varphi}_k \cdot \left|\Psi_{\omega^{-1}\lambda_0,\ldots,\omega\lambda_k,\ldots,\lambda_{n-1}}\right\rangle. \tag{21}$$

Furthermore, acting by (18) on $|\Psi_{\boldsymbol\lambda}\rangle$ and extracting the coefficient of $\lambda^{n+1}\mu^n$ we obtain

$$\mathbf{V}_n|\Psi_{\boldsymbol\lambda}\rangle = \left|\Psi_{\omega^{-1}\lambda_0,\lambda_1,\ldots,\lambda_{n-1}}\right\rangle. \tag{22}$$

Assuming generic parameters in L_k such that all proportionality factors are non-vanishing, by repeated application of $A_n(\lambda_k)$, $D_n(\lambda_k)$ and $\mathbf{V}_n$ to any eigenstate $|\Psi_{\boldsymbol\lambda}\rangle$ we span the whole N^n-dimensional space of states. Later, when we give explicit expressions for φ_k and $\widetilde{\varphi}_k$ we can check whether these factors can vanish.

So, if for a given set of parameters $a_k, b_k, c_k, d_k, \varkappa_k$, $(k = 1, \ldots, n)$ there is an eigenvector with the eigenvalue polynomial determined by the zeros $\boldsymbol\lambda$, then there are also eigenvectors to all eigenvalue polynomials determined by the zeros

$$\{\lambda_0\omega^{\rho_{n,1}}, \ldots, \lambda_{n-1}\omega^{\rho_{n,n-1}}\} \quad \text{with} \quad \boldsymbol\rho_n = (\rho_{n,0}, \ldots, \rho_{n,n-1}) \in (\mathbb{Z}_N)^n. \tag{23}$$

Let us therefore write the zeros as

$$\lambda_{n,k} = -r_{n,k}\omega^{\rho_{n,k}}, \tag{24}$$

where for n fixed, the n real numbers $r_{n,k}$ are determined by the $5n$ parameters $a_l, \ldots, \varkappa_l$. For fixed parameters, in all following calculations, we shall label the N^n eigenvectors by the $\boldsymbol\rho_n$ instead of our previous $\lambda_{n,k}$. For given parameters, the set of the eigenvalues is determined by the $r_{n,0}, \ldots, r_{n,n-1}$. The eigenvalue equation for $B_n(\lambda)$ becomes

$$B_n(\lambda)\left|\Psi_{\rho_n}\right\rangle = \lambda r_{n,0}\omega^{-\rho_{n,0}}\prod_{k=1}^{n-1}\left(\lambda + r_{n,k}\omega^{-\rho_{n,k}}\right)\left|\Psi_{\rho_n}\right\rangle. \tag{25}$$

In order to calculate the $r_{n,k}$ in terms of the parameters, we do not need the full quantum transfer matrix and the L_k-operators involving the Weyl variables. Rather, by the following averaging procedure [23]

$$\mathcal{O}(\lambda^N) = \langle O(\lambda^N)\rangle = \prod_{s\in\mathbb{Z}_N} O(\omega^s\lambda), \tag{26}$$

we associate with a spectral parameter dependent quantum operator $O(\lambda)$ a classical counterpart $\mathcal{O}(\lambda^N)$. We define the classical BSS model by the L-operator $\mathcal{L}_m(\lambda^N)$

$$\mathcal{L}_m(\lambda^N) = \begin{pmatrix}\langle L_{00}\rangle & \langle L_{01}\rangle\\ \langle L_{10}\rangle & \langle L_{11}\rangle\end{pmatrix} = \begin{pmatrix}1 - \epsilon\varkappa_m^N\lambda^N & -\epsilon\lambda^N\left(a_m^N - b_m^N\right)\\ c_m^N - d_m^N & b_m^N d_m^N/\varkappa_m^N - \epsilon\lambda^N a_m^N c_m^N\end{pmatrix}, \tag{27}$$

where $\epsilon = (-1)^N$. Analogously, we define the classical monodromy $\mathcal{T}_n$ by

$$\mathcal{T}_n = \mathcal{L}_1(\lambda^N)\mathcal{L}_2(\lambda^N)\cdots\mathcal{L}_n(\lambda^N) = \begin{pmatrix}\mathcal{A}_n(\lambda^N) & \mathcal{B}_n(\lambda^N)\\ \mathcal{C}_n(\lambda^N) & \mathcal{D}_n(\lambda^N)\end{pmatrix}. \tag{28}$$

Proposition 1.5 of [23] tells us that the classical polynomials $\mathcal{A}_n(\lambda^N), \mathcal{B}_n(\lambda^N), \mathcal{C}_n(\lambda^N)$ and $\mathcal{D}_n(\lambda^N)$ are the averages of their counterparts in (3): $\mathcal{A}_n(\lambda^N) = \langle A_n(\lambda)\rangle$, etc. So for $n \geqslant 2$ we have

$$\mathcal{B}_m(\lambda^N) = (-\epsilon)^m \lambda^N r_{m,0}^N \prod_{s=1}^{m-1} \left(\lambda^N - \epsilon r_{m,s}^N\right). \tag{29}$$

It is easy to derive [25] a three-term recursion which expresses $\mathcal{B}_m(\lambda^N)$ in terms of $\mathcal{B}_{m-1}(\lambda^N)$ and $\mathcal{B}_{m-2}(\lambda^N)$. Using the initial values $\mathcal{B}_1(\lambda^N) = -\epsilon\lambda^N r_1^N; \mathcal{B}_0(\lambda^N) = 0$ and defining $r_1^N = a_1^N - b_1^N$, this gives a $(n-1)$th degree algebraic relation for the $r_{m,s}^N$.

For the homogeneous model (the constants are taken to be site independent) this can be replaced by just a quadratic equation (see the appendix of [25]).

3.2. *Solving the auxiliary system: explicit construction of the eigenvectors of $B_n(\lambda)$*

The stepwise construction of the eigenvectors, starting with one site, then two site as linear combination of products of two one-site eigenvectors, etc is tedious because we have to go to four sites before the general rule emerges.

Let us start finding the one-site right eigenvectors $|\psi_\rho\rangle_1$ of $B_1(\lambda)$ as linear combination of spin states $|\gamma\rangle_1, \gamma \in \mathbb{Z}_N$, writing

$$|\psi_\rho\rangle_1 = \sum_{\gamma\in\mathbb{Z}_N} w_p(\gamma - \rho)|\gamma\rangle_1, \qquad \rho \in \mathbb{Z}_N. \tag{30}$$

Applying on the left B_1 from (1) and on the right (25), we demand

$$\lambda \mathbf{u}_1^{-1}(a_1 - b_1\mathbf{v}_1) \sum_{\gamma\in\mathbb{Z}_N} w_p(\gamma-\rho)|\gamma\rangle_1 = \lambda r_{1,0}\omega^{-\rho_{1,0}} \sum_{\gamma\in\mathbb{Z}_N} w_p(\gamma-\rho)|\gamma\rangle_1. \tag{31}$$

Applying (2) and shifting the left-hand summation for the term with $|\gamma+1\rangle_1$, we obtain

$$(a_1 - r_{1,0}\omega^{\gamma-\rho}) w_p(\gamma-\rho) = b_1 w_p(\gamma-\rho-1). \tag{32}$$

This is a difference equation for the function $w_p(\gamma)$ [44]:

$$\frac{w_p(\gamma)}{w_p(\gamma-1)} = \frac{y}{1-\omega^\gamma x}; \qquad w_p(0) = 1; \qquad \gamma \in \mathbb{Z}_N, \tag{33}$$

where we have put $y = b_1/a_1, r_{1,0} = x a_1$ and chose the initial value $w_p(0) = 1$. The cyclic property $w_p(\gamma) = w_p(\gamma+N)$ imposes the Fermat condition $x^N + y^N = 1$ on the two-component vector $p = (x, y)$. We indicate p as a subscript on the functions $w_p(\gamma)$. We shall consider the case of 'generic parameters', so in particular we exclude the case $a_k^N - b_k^N = 0$, and the 'superintegrable' case

$$a_k = \omega^{-1} b_k = c_k = d_k = \varkappa_k = 1, \tag{34}$$

since in the latter cases degenerations appear.

We write the analogous left eigenvector as

$${}_1\langle\psi_\rho| = \sum_{\gamma\in\mathbb{Z}_N} \frac{1}{w_p(\gamma-\rho-1)}\, {}_1\langle\gamma|, \qquad \rho \in \mathbb{Z}_N \tag{35}$$

with the same functions $w_p(\gamma)$, just now $p = (r_{1,0}/a_1, \omega^{-1}b_1/a_1)$. The Fermat vector-dependent functions $w_p(\gamma)$ play an important role for cyclic models. They are root-of-unity analogs of the q-gamma function.

By a similar calculation, the two-site eigenvectors are found to be

$$\left|\Psi_{\rho_{2,0},\rho_{2,1}}\right\rangle = \sum_{\rho_1,\rho_2\in\mathbb{Z}_N} \frac{\omega^{-(\rho_{2,0}+\rho_{2,1}-\rho_1)(\rho_{2,0}-\rho_2)}}{w_{p_{2,0}}(\rho_{2,0}-\rho_1-1)w_{\tilde{p}_2}(\rho_{2,0}+\rho_{2,1}-\rho_2-1)}|\psi_{\rho_1}\rangle_1\otimes|\psi_{\rho_2}\rangle_2, \qquad (36)$$

where $p_{2,0}=(x_{2,0},y_{2,0})$, $\tilde{p}_2=(\tilde{x}_2,\tilde{y}_2)$ and

$$x_{2,0}=a_2c_2\frac{r_1}{r_{2,0}}, \qquad y_{2,0}=\varkappa_1\frac{r_2}{r_{2,0}}, \qquad \tilde{x}_2=\frac{r_2}{r_{2,0}r_{2,1}}, \qquad \tilde{y}_2=\frac{b_2d_2}{\varkappa_2}\frac{r_1}{r_{2,0}r_{2,1}}. \qquad (37)$$

The condition that $p_{2,0}$ and $\tilde{p}_2$ are Fermat vectors determines $r_{2,0}$ and $r_{2,1}$.

The explicit formula for both the left- and right eigenvectors of $B_n(\lambda)$ for general number of sites n has been proved by lengthy induction and is given in [25]. A by-product of these calculations are the formulae for φ_k and $\widetilde{\varphi}_k$ introduced in (20) and (21):

$$A_n(\lambda_{n,k})\left|\Psi_{\boldsymbol{\rho}_n}\right\rangle=\varphi_k(\boldsymbol{\rho}_n')\left|\Psi_{\boldsymbol{\rho}_n^{+k}}\right\rangle, \qquad \varphi_k(\boldsymbol{\rho}_n')=-\frac{\tilde{r}_{n-1}}{r_n}\omega^{-\tilde{\rho}_n+\rho_{n,0}}F_n(\lambda_{n,k}/\omega)\prod_{s=1}^{n-2}y_{n-1,s}^{n,k}, \qquad (38)$$

$$D_n(\lambda_{n,k})\left|\Psi_{\boldsymbol{\rho}_n}\right\rangle=\tilde{\varphi}_k(\boldsymbol{\rho}_n')\left|\Psi_{\boldsymbol{\rho}_n^{+0,-k}}\right\rangle, \qquad \tilde{\varphi}_k(\boldsymbol{\rho}_n')=-\frac{r_n}{\tilde{r}_{n-1}}\frac{\omega^{\tilde{\rho}_n-\rho_{n,0}-1}}{\prod_{s=1}^{n-2}y_{n-1,s}^{n,k}}\prod_{m=1}^{n-1}F_m(\lambda_{n,k}), \qquad (39)$$

$$F_n(\lambda)=(b_n+\omega a_n\varkappa_n\lambda)(\lambda c_n+d_n/\varkappa_n). \qquad (40)$$

On the left of (38) and (39) the eigenvectors $\Psi_{\boldsymbol{\rho}_n}$ of $B_n(\lambda)$ are labeled by the vector

$$\boldsymbol{\rho}_n=(\rho_{n,0},\ldots,\rho_{n,n-1})\in(\mathbb{Z}_N)^n. \qquad (41)$$

$\boldsymbol{\rho}_n^{\pm k}$ denotes the vector $\boldsymbol{\rho}_n$ in which $\rho_{n,k}$ is replaced by $\rho_{n,k}\pm 1$:

$$\boldsymbol{\rho}_n^{\pm k}=(\rho_{n,0},\ldots,\rho_{n,k}\pm 1,\ldots,\rho_{n,n-1}), \qquad k=0,1,\ldots,n-1, \qquad (42)$$

$$\tilde{r}_n=r_{n,0}r_{n,1}\ldots r_{n,n-1} \qquad \text{and} \qquad \tilde{\rho}_n=\sum_{k=0}^{n-1}\rho_{n,k}. \qquad (43)$$

$\boldsymbol{\rho}_n'$ denotes the vector $\boldsymbol{\rho}_n$ without the component $\rho_{n,0}$:

$$\boldsymbol{\rho}_n'=(\rho_{n,1},\ldots,\rho_{n,n-1})\in(\mathbb{Z}_N)^{n-1}. \qquad (44)$$

The $y_{n-1,s}^{n,k}$ are components of a Fermat vector $p_{n-1,s}^{n,k}=\left(x_{n-1,s}^{n,k},y_{n-1,s}^{n,k}\right)$ defined by $x_{n-1,s}^{n,k}=r_{n,k}/r_{n-1,s}$ (see section 2.4 of [25]). The $F_m(\lambda)$ which appears in (38) and (39) is a factor of the quantum determinant:

$$A_n(\omega\lambda)D_n(\lambda)-C_n(\omega\lambda)B_n(\lambda)=\mathbf{V}_n\cdot\prod_{m=1}^{n}F_m(\lambda). \qquad (45)$$

From (1) we can read off directly the λ^0- and λ^n-coefficients of the polynomial $A_n(\lambda)$:

$$A_n(\lambda)=1+\cdots+\varkappa_1\varkappa_2\ldots\varkappa_n\mathbf{V}\lambda^n. \qquad (46)$$

Then using (38), the general action of $A_n(\lambda)$ on B_n eigenvectors can be written as an interpolation polynomial

$$A_n(\lambda)\left|\Psi_{\boldsymbol{\rho}_n}\right\rangle=\prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right)\left|\Psi_{\boldsymbol{\rho}_n}\right\rangle+\lambda\varkappa_1\cdots\varkappa_n\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\left|\Psi_{\boldsymbol{\rho}_n^{+0}}\right\rangle$$
$$+\sum_{k=1}^{n-1}\left(\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\lambda_{n,k}-\lambda_{n,s}}\right)\frac{\lambda}{\lambda_{n,k}}\varphi_k(\boldsymbol{\rho}_n')\left|\Psi_{\boldsymbol{\rho}_n^{+k}}\right\rangle. \qquad (47)$$

Considerable effort is needed to present the norm of an arbitrary state vector $|\Psi_{\rho_n}\rangle$ in factorized form, since multiple sums over the intermediate indices have to be performed. The norms are independent of the phase $\rho_{n,0}$ and their dependence on $\boldsymbol{\rho}'_n$ is

$$\langle\Psi_{\rho_n}|\Psi_{\rho_n}\rangle = \frac{C_n}{\prod_{l<m}(\lambda_{n,l}-\lambda_{n,m})} = \frac{C_n}{\prod_{l<m}(r_{n,m}\omega^{-\rho_{n,m}} - r_{n,l}\omega^{-\rho_{n,l}})}. \tag{48}$$

The normalizing factor C_n is independent of $\boldsymbol{\rho}_n$ and can be written recursively [26]. The two lowest values are

$$C_1 = \frac{N}{\omega}\left(\frac{x_1}{y_1}\right)^{N-1}, \qquad C_2 = C_1\frac{N^3}{\omega}\left(\frac{x_2}{y_2\tilde{y}_2 y_{2,0}}\right)^{N-1}. \tag{49}$$

3.3. Periodic model: Baxter equation and truncated functional equations

In the auxiliary problem we looked for eigenfunctions of B_n. B_n does not commute with $\mathbf{V}_n$ (8), see (22): $\mathbf{V}_n|\Psi_{\rho_n}\rangle = |\Psi_{\rho_n^{+0}}\rangle$. Now we are looking for eigenfunctions of $\mathbf{t}_n$ which commutes with $\mathbf{V}_n$. By Fourier transformation in $\rho_{n,0}$ we build a basis diagonal in $\mathbf{V}$, where the Fourier transformed variable $\rho\in\mathbb{Z}_N$ is the total $\mathbb{Z}_N$-charge:

$$|\tilde{\Psi}_{\rho,\rho'_n}\rangle = \sum_{\rho_{n,0}\in\mathbb{Z}_N}\omega^{-\rho\cdot\rho_{n,0}}|\Psi_{\rho_n}\rangle, \qquad \mathbf{V}_n|\tilde{\Psi}_{\rho,\rho'_n}\rangle = \omega^{\rho}|\tilde{\Psi}_{\rho,\rho'_n}\rangle. \tag{50}$$

We now write the eigenfunctions $|\Phi_{\rho,\mathbf{E}}\rangle$ of $\mathbf{t}_n(\lambda)$ as linear combination of the $|\tilde{\Psi}_{\rho,\rho'_n}\rangle$. The eigenvalues of $\mathbf{t}_n(\lambda)$ on these states are again order n polynomials in λ:

$$\mathbf{t}_n(\lambda)|\Phi_{\rho,\mathbf{E}}\rangle = (E_0 + E_1\lambda + \cdots + E_{n-1}\lambda^{n-1} + E_n\lambda^n)|\Phi_{\rho,\mathbf{E}}\rangle. \tag{51}$$

Since the values of E_0 and E_n can be read off immediately from (8):

$$E_0 = 1 + \omega^{\rho}\prod_{m=1}^{n} b_m d_m/\varkappa_m, \qquad E_n = \prod_{m=1}^{n} a_m c_m + \omega^{\rho}\prod_{m=1}^{n}\varkappa_m, \tag{52}$$

we combine the remaining coefficients into a vector $\mathbf{E} = \{E_1,\ldots,E_{n-1}\}$, and label the eigenvectors just by the charge ρ and $\mathbf{E}$:

$$\mathbf{t}_n(\lambda)|\Phi_{\rho,\mathbf{E}}\rangle = t_n(\lambda|\rho,\mathbf{E})|\Phi_{\rho,\mathbf{E}}\rangle, \qquad |\Phi_{\rho,\mathbf{E}}\rangle = \sum_{\rho'_n}\mathcal{Q}^{\mathrm{R}}(\boldsymbol{\rho}'_n|\rho,\mathbf{E})|\tilde{\Psi}_{\rho,\rho'_n}\rangle. \tag{53}$$

Now, in order to achieve SoV of the multi-variable functions $\mathcal{Q}^{\mathrm{R}}$, we split off from $\mathcal{Q}^{\mathrm{R}}(\boldsymbol{\rho}'_n|\rho,\mathbf{E})$ Sklyanin's separating factor:

$$\mathcal{Q}^{\mathrm{R}}(\boldsymbol{\rho}'_n|\rho,\mathbf{E}) = \frac{\prod_{k=1}^{n-1}Q_k^{\mathrm{R}}(\rho_{n,k})}{\prod_{\substack{s,s'=1\\(s\neq s')}}^{n-1} w_{p_{n,s}^{n,s'}}(\rho_{n,s}-\rho_{n,s'})}. \tag{54}$$

We shall not give the detailed calculation and just indicate the main mechanism. We express $\mathbf{t}_n(\lambda)$ as an interpolation polynomial through the zeros $\lambda_{n,k}$ of $B_n(\lambda)$:

$$\begin{aligned}(A_n(\lambda)+D_n(\lambda))|\tilde{\Psi}_{\rho,\rho'_n}\rangle &= \left\{E_0\prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right) + \lambda E_n\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\right\}\tilde{\Psi}_{\rho,\rho'_n}\\ &+\sum_{k=1}^{n-1}\left(\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\lambda_{n,k}-\lambda_{n,s}}\right)\frac{\lambda}{\lambda_{n,k}}\left(\varphi_k(\boldsymbol{\rho}'_n)|\tilde{\Psi}_{\rho,\rho'^{+k}_n}\rangle + \omega^{\rho}\tilde{\varphi}_k(\boldsymbol{\rho}'_n)|\tilde{\Psi}_{\rho,\rho'^{-k}_n}\rangle\right).\end{aligned} \tag{55}$$

When we evaluate (55) successively at the $n-1$ values $\lambda = \lambda_{n,k}$, $k = 1,\ldots,n-1$, the terms on the right of the first line of (55) do not contribute. Due to the Sklyanin-factor the brackets involving the differences $\lambda_{n,k} - \lambda_{n,s}$ are made to cancel, leading to SoV. This results in the $n-1$ single-variable $\lambda_{n,k}$ Baxter equations ($k = 1, \ldots, n-1$)

$$t_n(\lambda_{n,k}|\rho, \mathbf{E})\, Q_k^{\mathrm{R}}(\rho_{n,k}) = \Delta_k^{+}(\lambda_{n,k})\, Q_k^{\mathrm{R}}(\rho_{n,k}+1) + \Delta_k^{-}(\omega\lambda_{n,k})\, Q_k^{\mathrm{R}}(\rho_{n,k}-1). \tag{56}$$

Starting from the left eigenvectors the analogous left Baxter equations are

$$t_n(\lambda_{n,k}|\rho, \mathbf{E})\, Q_k^{\mathrm{L}}(\rho_{n,k}) = \omega^{n-1}\Delta_k^{-}(\lambda_{n,k})\, Q_k^{\mathrm{L}}(\rho_{n,k}+1) + \omega^{1-n}\Delta_k^{+}(\omega\lambda_{n,k})\, Q_k^{\mathrm{L}}(\rho_{n,k}-1), \tag{57}$$

where we abbreviated

$$\Delta_k^{+}(\lambda) = (\omega^{\rho}/\chi_k)(\lambda/\omega)^{1-n}\prod_{m=1}^{n-1} F_m(\lambda/\omega), \qquad \Delta_k^{-}(\lambda) = \chi_k(\lambda/\omega)^{n-1} F_n(\lambda/\omega). \tag{58}$$

χ_k collects several factors (partly arising from φ_k and $\widetilde{\varphi}_k$) determined by constants $\varkappa_k, a_k, \ldots, d_k$ alone. Now note that the left-hand side of (56) more explicitly reads

$$\left(E_0 + \sum_{s=1}^{n-1} E_s \lambda_{n,k}^{s} + E_n \lambda_{n,k}^{n}\right) Q_k^{\mathrm{R}}(\rho_{n,k}) = \cdots$$

where the $\mathbf{E}$ are unknown and have to be determined from the system of homogeneous equations (56) together with the $n-1$ functions $Q_k^{\mathrm{R}}(\rho_{n,k})$. In order to have a non-trivial solution, the coefficient determinants have to be degenerate. Fix a k, and then from the determinant we may get one relation among $E_0, \ldots, E_n$. All $n-1$ systems for different k should be sufficient to determine all components of $\mathbf{E}$. Fortunately, the condition for non-trivial solutions to (56) can be written as well-known truncated functional equations:

Define $\tau^{(2)}(\lambda) = t(\lambda)$ [5] and construct a fusion hierarchy [45, 33] by setting $\tau^{(0)}(\lambda) = 0$, $\tau^{(1)}(\lambda) = 1$, and

$$\tau^{(j+1)}(\lambda) = \tau^{(2)}(\omega^{j-1}\lambda)\tau^{(j)}(\lambda) - \omega^{\rho} z(\omega^{j-1}\lambda)\tau^{(j-1)}(\lambda), \qquad j = 2, 3, \ldots, N, \tag{59}$$

where

$$z(\lambda) = \omega^{-\rho}\Delta^{+}(\lambda)\Delta^{-}(\lambda) = \prod_{m=1}^{n} F_m(\lambda/\omega). \tag{60}$$

Then it can be shown [25] that if $\tau^{(N+1)}(\lambda)$ satisfies the truncation identity

$$\tau^{(N+1)}(\lambda) - \omega^{\rho} z(\lambda)\tau^{(N-1)}(\omega\lambda) = \mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N) \tag{61}$$

with $\mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N)$ given in (28), then the system (56) has a non-trivial solution for all k. This truncated hierarchy can be used to find the transfer-matrix eigenvalues [46, 47]. In our construction we have even more: for every solution of (59) and (61) we can construct an eigenvector.

4. Action of $\mathbf{u}_k$ and $\mathbf{v}_k$ on eigenstates of $B_n(\lambda)$

Our main aim is to calculate matrix elements of the local operators $\mathbf{u}_k$ and $\mathbf{v}_k$ between eigenstates $|\Phi_{\rho,\mathbf{E}}\rangle$ of $\mathbf{t}_n(\lambda)$. Since we know how to get these states from the $B_n(\lambda)$ eigenstates (53)–(56), we first set out to find the action of the local operators on the $|\Psi_{\rho_n}\rangle$. Since we built our auxiliary states successively from one-site to n-site, the formulae will not be symmetric between, e.g. $\mathbf{u}_j$ and $\mathbf{u}_k$ with $j \neq k$.

For $\mathbf{u}_n$ we can calculate its action directly. Starting from

$$\mathbf{u}_n^{-1}(a_n - b_n\mathbf{v}_n)|\psi_{\rho_n}\rangle_n = r_n\omega^{-\rho_n}|\psi_{\rho_n}\rangle_n,$$

[5] This definition in [29] is the origin of calling the BBS model the $\tau^{(2)}$-model.

we get the formula for the action of $\mathbf{u}_n$ on one-site eigenvectors:

$$\mathbf{u}_n|\psi_{\rho_n}\rangle_n = \frac{\omega^{\rho_n}}{r_n}\left(a_n|\psi_{\rho_n}\rangle_n - b_n|\psi_{\rho_n+1}\rangle_n\right). \tag{62}$$

Using then the explicit recursion formula relating $|\Psi_{\rho_n}\rangle$ to $|\Psi_{\rho_{n-1}}\rangle$ one finds [26]:

$$\mathbf{u}_n|\Psi_{\rho_n}\rangle = \frac{a_n}{\tilde{r}_n\omega^{-\tilde{\rho}_n}}|\Psi_{\rho_n}\rangle - \frac{b_n\varkappa_1\varkappa_2\cdots\varkappa_{n-1}}{r_{n,0}\omega^{-\rho_{n,0}}}|\Psi_{\rho_n^{+0}}\rangle$$
$$+\sum_{k=1}^{n-1}\frac{a_nb_n\varphi_k(\rho_n')}{r_{n,0}\omega^{-\rho_{n,0}}\lambda_{n,k}(b_n+a_n\varkappa_n\lambda_{n,k})\prod_{s\neq k}(\lambda_{n,k}-\lambda_{n,s})}|\Psi_{\rho_n^{+k}}\rangle. \tag{63}$$

We shall derive this result in a simpler way expressing the local operators $\mathbf{u}_k$ and $\mathbf{v}_k$ in terms of the global entries A_n and B_n of the monodromy matrix, taken at particular values of λ. There is a well-known method elaborated by the Lyon group [48]. However, this method requires the fulfillment of the condition $R(0) = P$ with R being the quantum R-matrix intertwining two L-operators in quantum spaces and P the permutation operator. This requirement is fulfilled for the cyclic L-operators only at special values of parameters where the R-matrix is the product of four weights of the Chiral Potts model [29]. Another requirement regards the possibility to obtain such a R-matrix by fusion in the auxiliary space of the initial L-operator. This requirement cannot be fulfilled for the cyclic L-operators (1) because the fusion in the auxiliary space [49] gives L-operators with the highest weight evaluation representations of the corresponding quantum affine algebra, but we need cyclic-type representation in the auxiliary space.

We will use an idea borrowed from a paper of Kuznetsov on SoV for classical systems [50]. What we can do is the following: Consider the inverse of the operator $L_k(\lambda)$:

$$L_k^{-1}(\lambda) = \begin{pmatrix}\omega\lambda a_kc_k+\mathbf{v}_kb_kd_k/\varkappa_k & -\lambda\mathbf{u}_k^{-1}(a_k-b_k\mathbf{v}_k)\\ -\omega\mathbf{u}_k(c_k-d_k\mathbf{v}_k), & 1+\omega\lambda\varkappa_k\mathbf{v}_k\end{pmatrix}\cdot(\det{}_qL_k(\lambda))^{-1}, \tag{64}$$

where

$$\det{}_qL_k(\lambda) = \mathbf{v}_kF_k(\lambda), \qquad F_k(\lambda) = (b_k+\omega\lambda a_k\varkappa_k)(\lambda c_k+d_k/\varkappa_k).$$

The expression for $L_k^{-1}(\lambda)$ is singular at zeros $\lambda_k' = -b_k/(\omega a_k\varkappa_k)$ and $\lambda_k'' = -d_k/(c_k\varkappa_k)$ of $F_k(\lambda)$. Of course,

$$T_{n-1}(\lambda) = T_n(\lambda)L_n^{-1}(\lambda). \tag{65}$$

Therefore at the zeros of $F_n(\lambda)$ the left-hand side is regular in λ and the right-hand side also has to be regular. At $\lambda = \lambda_n' = -b_n/(\omega a_n\varkappa_n)$ we obtain

$$A_n(\lambda_n')\mathbf{u}_n^{-1}b_n/(\omega\varkappa_n) + B_n(\lambda_n') = 0.$$

Hence we have a formula for $\mathbf{u}_n$:

$$\mathbf{u}_n = \lambda_n'a_nB_n^{-1}(\lambda_n')A_n(\lambda_n'). \tag{66}$$

From the condition of the regularity of the right-hand side of (65) at $\lambda = \lambda_n'' = -d_n/(c_n\varkappa_n)$ we obtain

$$A_n(\lambda_n'')(-\lambda_n'')\mathbf{u}_n^{-1}(a_n-b_n\mathbf{v}_n) + B_n(\lambda_n'')(1-d_n/(\omega c_n)\mathbf{v}_n) = 0.$$

Excluding $\mathbf{u}_n$ by means of (66), we obtain the formula for $\mathbf{v}_n$:

$$\mathbf{v}_n = -1/(\omega\varkappa_n)(A_n(\lambda_n')B_n(\lambda_n'') - A_n(\lambda_n'')B_n(\lambda_n'))^{-1}$$
$$\times(A_n(\lambda_n')B_n(\lambda_n'')/\lambda_n'' - A_n(\lambda_n'')B_n(\lambda_n')/\lambda_n'). \tag{67}$$

Using the RTT-relations following from (5), we can permute A_n and B_n^{-1} in (66) to get an equivalent formula

$$\mathbf{u}_n = \omega\lambda_n' a_n A_n(\omega\lambda_n') B_n^{-1}(\omega\lambda_n'). \tag{68}$$

Using (47) and (25) we obtain (63).

We can also get the formulas for $\mathbf{u}_{n-1}$ and $\mathbf{v}_{n-1}$. We express $L_n^{-1}(\lambda)$ in terms of $A_n(\lambda)$ and $B_n(\lambda)$ using (66) and (67). Now the formula (65) allows one to find expressions for $A_{n-1}(\lambda)$ and $B_{n-1}(\lambda)$ in terms of $A_n(\lambda)$ and $B_n(\lambda)$. Finally we substitute these expressions into (66) and (67) in which the indices n are replaced by $n-1$. This gives us expressions for $\mathbf{u}_{n-1}$ and $\mathbf{v}_{n-1}$ in terms of $A_n(\lambda)$ and $B_n(\lambda)$. The described procedure can be iterated to express the local operators $\mathbf{u}_k$ and $\mathbf{v}_k$ in terms of $A_n(\lambda)$ and $B_n(\lambda)$. For example, the result for $\mathbf{u}_{n-1}$ is

$$\mathbf{u}_{n-1} = \omega\lambda_{n-1}' a_{n-1}(A_n(\omega\lambda_{n-1}')(\omega^2\lambda_{n-1}' a_n c_n + \mathbf{v}_n b_n d_n/\varkappa_n) - B_n(\omega\lambda_{n-1}')\omega\mathbf{u}_n(c_n - d_n\mathbf{v}_n))$$
$$\times\left(-A_n(\omega\lambda_{n-1}')\omega\lambda_{n-1}'\mathbf{u}_n^{-1}(a_n - b_n\mathbf{v}_n) + B_n(\omega\lambda_{n-1}')(1+\omega^2\lambda_{n-1}'\varkappa_n\mathbf{v}_n)\right)^{-1},$$

where $\omega\lambda_{n-1}' = -b_{n-1}/(a_{n-1}\varkappa_{n-1})$ and expressions (68) and (67) for $\mathbf{u}_n$ and $\mathbf{v}_n$ have to be substituted. It gives the action of $\mathbf{u}_{n-1}$ on $\left|\Psi_{\rho_n}\right\rangle$. We see that the formula gets quite involved. However, $\mathbf{u}_1$ can easily be expressed in terms of D_n and B_n:

$$\mathbf{u}_1 = \frac{1}{c_1} D_n\left(-\frac{d_1}{c_1\varkappa_1}\right) B_n^{-1}\left(-\frac{d_1}{c_1\varkappa_1}\right). \tag{69}$$

For our purpose of finding matrix elements of spin operator between eigenstates $|\Phi_{\rho,\mathbf{E}}\rangle$ of homogeneous $\mathbf{t}_n(\lambda)$ we can choose any spin operator $\mathbf{u}_k$ because they all are related by the action of translation operator having the same eigenstates $|\Phi_{\rho,\mathbf{E}}\rangle$. In what follows we consider matrix elements of the spin operator $\mathbf{u}_n$ because the corresponding formula for the action (63) is the simplest.

At the end of this section we would like to mention some similarity of our formulae with the formulae from the paper [51], where the local operators for the quantum Toda chain are expressed in terms of quantum separated variables with the use of a recursive construction of the eigenvectors [22].

5. The general inhomogeneous $N = 2$ BBS model

In the $N = 2$ case we have two charge sectors $\rho = 0, 1$. Following the language of e.g. [14, 17, 24] the sector $\rho = 0$ will be called the Neveu–Schwarz (NS)-sector, and $\rho = 1$ the Ramond (R) sector. We are going to show that the spin matrix elements can be written in a fairly compact, although not yet factorized form (85) and (86). The full factorization will be achieved later for the homogeneous Ising case.

5.1. Solving the Baxter equations and norm of states

Let us fix an eigenvalue polynomial $t(\lambda|\rho,\mathbf{E})$ of $\mathbf{t}(\lambda)$ corresponding to a right eigenvector $|\Phi_{\rho,\mathbf{E}}\rangle$ (since in the following our chain will have the fixed length n we often shall skip the index n. Also sometimes we shall suppress the arguments $\rho, \mathbf{E}$ in t).

In order to find $|\Phi_{\rho,\mathbf{E}}\rangle$ explicitly we have to solve the associated $n-1$ systems ($k = 1, 2, \ldots, n-1$) of (right) Baxter equations:

$$\begin{aligned} t(-r_{n,k})Q_k^{\mathrm{R}}(0) &= \left(\Delta_k^+(-r_{n,k}) + \Delta_k^-(r_{n,k})\right) Q_k^{\mathrm{R}}(1),\\ t(r_{n,k})Q_k^{\mathrm{R}}(1) &= \left(\Delta_k^+(r_{n,k}) + \Delta_k^-(-r_{n,k})\right) Q_k^{\mathrm{R}}(0). \end{aligned} \tag{70}$$

Since $t(\lambda|\rho,\mathbf{E})$ is eigenvalue polynomial, the functional relation (61) ensures the existence of non-trivial solutions to (70) with respect to the unknown variables $Q_k^{\mathrm{R}}(0)$ and $Q_k^{\mathrm{R}}(1)$ for every

$k = 1, 2, \ldots, n-1$. In the $N = 2$ case, this means that for every k we have one independent linear equation (in the case of degenerate eigenvalues, possibly no equation). In the case of generic parameters, both sides of each equation will be non-zero. So, fixing $Q_k^{\mathrm{R}}(0) = 1$ we obtain two equivalent expressions for $Q_k^{\mathrm{R}}(1)$:

$$Q_k^{\mathrm{R}}(1) = \frac{t(-r_{n,k})}{\Delta_k^+(-r_{n,k}) + \Delta_k^-(r_{n,k})} = \frac{\Delta_k^+(r_{n,k}) + \Delta_k^-(-r_{n,k})}{t(r_{n,k})}. \tag{71}$$

Analogously from the left-Baxter equations, fixing $Q_k^{\mathrm{L}}(0) = 1$ we obtain

$$Q_k^{\mathrm{L}}(1) = \frac{(-1)^{n-1} t(-r_{n,k})}{\Delta_k^+(r_{n,k}) + \Delta_k^-(-r_{n,k})} = \frac{\Delta_k^+(-r_{n,k}) + \Delta_k^-(r_{n,k})}{(-1)^{n-1} t(r_{n,k})}.$$

Since for generic parameters $t(r_{n,k}|\rho, \mathbf{E}) \neq 0$ these explicit formulae give

$$Q_k^{\mathrm{L}}(\rho_{n,k}) Q_k^{\mathrm{R}}(\rho_{n,k}) = (-1)^{\rho_{n,k}(n-1)} t((-1)^{\rho_{n,k}} r_{n,k})/t(r_{n,k}).$$

To get the periodic state, we have to insert the Skylanin-separation factor (54). Now for $N = 2$ the functions w_p are simple:

$$w_p(0) = 1, \qquad w_p(1) = \frac{y}{1+x} = \frac{1-x}{y}, \qquad (w_p(1))^2 = \frac{1-x}{1+x}. \tag{72}$$

In the Sklyanin factor we have to use the Fermat point $p_{n,l}^{n,m} = \left(x_{n,l}^{n,m}, y_{n,l}^{n,m}\right)$ defined by the coordinate $x_{n,l}^{n,m} = r_{n,m}/r_{n,l}$. Here it can be expressed it in terms of $x_{n,l}^{n,m}$ only and we obtain

$$\frac{\langle \Phi_{\rho,\mathbf{E}} | \Phi_{\rho,\mathbf{E}} \rangle}{\langle \tilde{\Psi}_{\rho,\rho_n'} | \tilde{\Psi}_{\rho,\rho_n'} \rangle} = \sum_{\rho_n'} \frac{\prod_{l<m}^{n-1} (-1)^{\rho_{n,l}+\rho_{n,m}} (r_{n,m} + r_{n,l})^2 \prod_{k=1}^{n-1} Q_k^{\mathrm{L}}(\rho_{n,k}) Q_k^{\mathrm{R}}(\rho_{n,k})}{\prod_{l<m}^{n-1} ((-1)^{\rho_{n,l}} r_{n,l} + (-1)^{\rho_{n,m}} r_{n,m})^2}. \tag{73}$$

We can normalize to a convenient reference state. For the moment, simple formulae arise if for the normalization we chose the auxiliary state $|\tilde{\Psi}_{0,\mathbf{0}}\rangle$ where $\mathbf{0} = (0, 0, \ldots, 0)$. From (48) we obtain

$$\frac{\langle \tilde{\Psi}_{\rho,\rho_n'} | \tilde{\Psi}_{\rho,\rho_n'} \rangle}{\langle \tilde{\Psi}_{0,\mathbf{0}} | \tilde{\Psi}_{0,\mathbf{0}} \rangle} = \frac{\prod_{l<m}^{n-1} (r_{n,m}(-1)^{\rho_{n,m}} + r_{n,l}(-1)^{\rho_{n,l}})}{\prod_{l<m}^{n-1} (r_{n,m} + r_{n,l})}. \tag{74}$$

Combining all these formulae we get for the left-right overlap of the transfer-matrix eigenvectors of the periodic BBS model at $N = 2$:

$$\frac{\langle \Phi_{\rho,\mathbf{E}} | \Phi_{\rho,\mathbf{E}} \rangle}{\langle \tilde{\Psi}_{0,\mathbf{0}} | \tilde{\Psi}_{0,\mathbf{0}} \rangle} = \frac{\prod_{l<m}^{n-1} (r_{n,m} + r_{n,l})}{\prod_{l=1}^{n-1} t(r_{n,l})} \sum_{\rho_n'} \frac{\prod_{l=1}^{n-1} (-1)^{\rho_{n,l}} t((-1)^{\rho_{n,l}} r_{n,l})}{\prod_{l<m}^{n-1} ((-1)^{\rho_{n,m}} r_{n,m} + (-1)^{\rho_{n,l}} r_{n,l})}. \tag{75}$$

This formula is not yet very useful since from (53) it contains the summation over the $n-1$ $\mathbf{Z}_2$-variables ρ_n' defined in (44). However, in [26] it is shown how to perform this sum explicitly, and the fully factorized result is

$$\frac{\langle \Phi_{\rho,\mathbf{E}} | \Phi_{\rho,\mathbf{E}} \rangle}{\langle \tilde{\Psi}_{0,\mathbf{0}} | \tilde{\Psi}_{0,\mathbf{0}} \rangle} = 2^{n-1} \tilde{r}_n' \frac{\prod_{l<m}^{n-1} (r_{n,m} + r_{n,l})}{\prod_{k=1}^{n} \prod_{l=1}^{n-1} (r_{n,l} + \mu_k)} \prod_{i<j}^{n} (\mu_i + \mu_j), \tag{76}$$

where $-\mu_i$ are the zeros of the eigenvalue polynomial of $\mathbf{t}(\lambda|\rho, \mathbf{E})$:

$$\mathbf{t}(\lambda|\rho, \mathbf{E}) |\Phi_{\rho,\mathbf{E}}\rangle = \Lambda \prod_{i=1}^{n} (\lambda + \mu_i) |\Phi_{\rho,\mathbf{E}}\rangle. \tag{77}$$

We do not specify the factor Λ, since in the following it will cancel.

5.2. Matrix elements between eigenvectors of the periodic $N = 2$ BBS model

In (63) we obtained the action of $\mathbf{u}_n$ on an eigenvector $\left|\Psi_{\rho_n}\right\rangle$ of $B_n(\lambda)$: the result is a linear combination of the original vector plus a sum of vectors which each have one component of $\boldsymbol{\rho}_n$ shifted. In order to get the matrix elements of $\mathbf{u}_n$ in the periodic model, using (50) we first pass to charge eigenstates $\left\langle\tilde{\Psi}_{\rho,\rho'_n}\right|, \left|\tilde{\Psi}_{\rho,\rho'_n}\right\rangle$:

$$\left\langle\tilde{\Psi}_{\rho,\rho'_n}\right| = \left\langle\Psi_{0,\rho'_n}\right| + (-)^\rho\left\langle\Psi_{1,\rho'_n}\right|, \qquad \left|\tilde{\Psi}_{\rho,\rho'_n}\right\rangle = \left|\Psi_{0,\rho'_n}\right\rangle + (-)^\rho\left|\Psi_{1,\rho'_n}\right\rangle. \tag{78}$$

Since $\omega = -1$, $\mathbf{u}_n$ anti-commutes with $\mathbf{V}_n$ so that only matrix elements of $\mathbf{u}_n$ between states of different charge ρ can be nonzero. In the following we shall chose the right eigenvector from $\rho = 1$, and then the left eigenvector must have $\rho = 0$ (the opposite choice gives a different sign in (79)). Using (63), we find

$$\frac{\left\langle\tilde{\Psi}_{0,\rho'_n}\middle|\mathbf{u}_n\middle|\tilde{\Psi}_{1,\rho'_n}\right\rangle}{\left\langle\tilde{\Psi}_{0,\rho'_n}\middle|\tilde{\Psi}_{0,\rho'_n}\right\rangle} = \frac{a_n}{\tilde{r}_n}(-1)^{\tilde{\rho}'_n} - \frac{\varkappa_1\varkappa_2\cdots\varkappa_{n-1}b_n}{r_{n,0}}, \tag{79}$$

$$\frac{\left\langle\tilde{\Psi}_{0,\rho'^{+k}_n}\middle|\mathbf{u}_n\middle|\tilde{\Psi}_{1,\rho'_n}\right\rangle}{\left\langle\tilde{\Psi}_{0,\rho'_n}\middle|\tilde{\Psi}_{0,\rho'_n}\right\rangle} = \frac{\tilde{r}_{n-1}a_nb_nc_n}{r_nr_{n,0}}\left(1 + \frac{(-1)^{\rho_{n,k}}d_n}{\varkappa_nc_nr_{n,k}}\right)\frac{(-1)^{\tilde{\rho}'_n}\prod_{l=1}^{n-2}y^{n,k}_{n-1,l}}{\prod_{s\neq k}(r_{n,k}(-1)^{\rho_{n,k}} + r_{n,s}(-1)^{\rho_{n,s}})}. \tag{80}$$

Of physical interest are the matrix elements between periodic eigenstates. To get these we have to form linear combinations determined by the solutions of the Baxter equations: Recall (53): $|\Phi_{\rho,\mathbf{E}}\rangle = \sum_{\rho'_n}\mathcal{Q}^{\mathrm{R}}(\rho'_n|\rho,\mathbf{E})\left|\tilde{\Psi}_{\rho,\rho'_n}\right\rangle$ and the corresponding left equations.

Let $\langle\Phi_0|$ be a left eigenvector of the transfer-matrix $\mathbf{t}_n(\lambda)$ with $\rho = 0$ and $|\Phi_1\rangle$ be a right eigenvector with $\rho = 1$ (often suppressing the subscripts $\mathbf{E}, \mathbf{E}'$):

$$\langle\Phi_{0,\mathbf{E}'}|\mathbf{t}(\lambda|0,\mathbf{E}') = t^{(0)}(\lambda)\langle\Phi'_{0,\mathbf{E}}|, \qquad \mathbf{t}(\lambda|1,\mathbf{E})|\Phi_{1,\mathbf{E}}\rangle = t^{(1)}(\lambda)|\Phi_{1,\mathbf{E}}\rangle. \tag{81}$$

Let $Q^{\mathrm{L}(0)}_k(\rho_{n,k})$ and $Q^{\mathrm{R}(1)}_k(\rho_{n,k})$ be the solutions of Baxter equation corresponding to these two eigenvectors. After some simplification we get for the matrix elements (keeping the normalization by the auxiliary 'reference' state):

$$\frac{\langle\Phi_0|\sigma^z_n|\Phi_1\rangle}{\langle\tilde{\Psi}_{0,\mathbf{0}}|\tilde{\Psi}_{0,\mathbf{0}}\rangle} = \sum_{\boldsymbol{\rho}'}\mathcal{N}(\boldsymbol{\rho}')\left(R_0(\boldsymbol{\rho}')\left(\frac{a_n}{\tilde{r}}(-1)^{\tilde{\rho}'} - \frac{\varkappa_1\varkappa_2\cdots\varkappa_{n-1}b_n}{r_0}\right) + \sum_{k=1}^{n-1}R_k(\boldsymbol{\rho}')\right), \tag{82}$$

where

$$\mathcal{N}(\boldsymbol{\rho}') = (-1)^{n\tilde{\rho}'}\prod_{l<m}^{n-1}\frac{r_l + r_m}{r_l(-1)^{\rho_l} + r_m(-1)^{\rho_m}}, \qquad R_0(\boldsymbol{\rho}') = \prod_{l=1}^{n-1}Q^{\mathrm{L}(0)}_l(\rho_l)Q^{\mathrm{R}(1)}_l(\rho_l), \tag{83}$$

$$R_k(\boldsymbol{\rho}') = -\frac{a_nb_nc_n}{r_0}Q^{\mathrm{L}(0)}_k(\rho_k+1)Q^{\mathrm{R}(1)}_k(\rho_k)\prod_{l\neq k}^{n-1}Q^{\mathrm{L}(0)}_l(\rho_l)Q^{\mathrm{R}(1)}_l(\rho_l)$$
$$\times\left(1 - \frac{d_n}{\varkappa_nc_nv_k}\right)\frac{v_k^{n-1}\chi_k}{\prod_{s\neq k}(v_k - v_s)} \tag{84}$$

with $r_k = r_{n,k}$, $\rho_k = \rho_{n,k}$, $k = 0, 1, \ldots, n-1$,

$$v_k = -r_k(-1)^{\rho_k}, \qquad \tilde{r} = r_0r_1\cdots r_{n-1} \qquad \text{and} \qquad \tilde{\rho}' = \sum_{k=1}^{n-1}\rho_k.$$

The origin of the different terms in (82) is as follows: the sum over $\boldsymbol{\rho}'$ comes from (53), $\mathcal{N}(\boldsymbol{\rho}')$ is the normalization factor from (74). The terms at $R_0(\boldsymbol{\rho}')$ arise from the first line of (63): the shift in $\rho_{n,0}$ affects the charge sector only. The sum over k and expression for $R_k(\rho')$ come from the second line in (63). Now, the sum over k can be performed. Indeed, as shown in [27], using the Baxter equations, some cancellations take place and (82) can be written as

$$\frac{\langle\Phi_0|\mathbf{u}_n|\Phi_1\rangle}{\langle\tilde{\Psi}_{0,\mathbf{0}}|\tilde{\Psi}_{0,\mathbf{0}}\rangle} = \frac{a_n}{2r_0}\sum_{\rho'\in\mathbb{Z}_2^{n-1}} \mathcal{N}(\rho')R_0(\rho')R(\rho') \tag{85}$$

with

$$R(\rho') = \frac{t^{(0)}(-\zeta_n)}{\prod_{l=1}^{n-1}(-\zeta_n+(-1)^{\rho_l}r_l)} + \frac{t^{(1)}(\zeta_n)}{\prod_{l=1}^{n-1}(\zeta_n+(-1)^{\rho_l}r_l)}, \qquad \zeta_n = \frac{b_n}{a_n\varkappa_n}. \tag{86}$$

Despite the simple appearance, for the general inhomogeneous $N = 2$ BBS model, performing the sums over the $\mathbb{Z}_2$ variables explicitly seems to be a presently hopeless task. However, for the homogeneous Ising model we shall show this to be possible.

6. Homogeneous $N = 2$ BBS model

6.1. Spectra and zeros of the B_n- and t_n-eigenvalue polynomials

We now specialize to $N = 2$ and take all parameters site independent (‘homogeneous’):

$$a_m = a, \quad b_m = b, \quad c_m = c, \quad d_m = d, \quad \varkappa_m = \varkappa, \quad r_m = r, \quad \mathcal{L}_m(\lambda^2) = \mathcal{L}(\lambda^2), \quad \forall m. \tag{87}$$

Then the classical monodromy is

$$\begin{pmatrix} \mathcal{A}_n(\lambda^2) & \mathcal{B}_n(\lambda^2) \\ \mathcal{C}_n(\lambda^2) & \mathcal{D}_n(\lambda^2) \end{pmatrix} = (\mathcal{L}(\lambda^N))^n. \tag{88}$$

Consider trace, determinant and eigenvalues $x_\pm$ of $\mathcal{L}$:

$$\tau(\lambda^2) = \operatorname{tr}\mathcal{L}(\lambda^2) = 1 + \frac{b^2d^2}{\varkappa^2} - \lambda^2(\varkappa^2 + a^2c^2), \tag{89}$$

$$\delta(\lambda^2) = \det\mathcal{L}(\lambda^2) = (b^2/\varkappa^2 - \lambda^2a^2)(d^2 - \lambda^2c^2\varkappa^2) = F(\lambda)F(-\lambda), \tag{90}$$

$$x_\pm = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\delta}\right), \qquad F(\lambda) = (b - a\varkappa\lambda)(\lambda c + d/\varkappa). \tag{91}$$

From the matrix $\mathcal{L}(\lambda^2)$ we obtain

$$\mathcal{B}_m(\lambda^2) = -\lambda^2(a^2 - b^2)\left(x_+^n - x_-^n\right)/(x_+ - x_-), \tag{92}$$

so that the zeros of $\mathcal{B}_m$ are at $x_+/x_- = \mathrm{e}^{\mathrm{i}m\phi_{n,s}}$ with

$$\phi_{n,s} = 2\pi s/n, \qquad s = 1, 2, \ldots, n-1, \qquad s \neq 0. \tag{93}$$

Using $\tau^2 = 4\delta\cos^2(\phi/2)$ and (89), (90) we can translate the zeros labeled by $\phi_{n,s}$ by a quadratic equation in λ^2 into zeros $\lambda_{n,s}$.

Now we solve the functional relations (59) and (61) for the transfer matrix spectrum. Using (59) for $j = 2$ and eliminating $\tau^{(3)}$ by (61) we get the functional relation

$$t(\lambda)t(-\lambda) = (-1)^\rho(z(\lambda) + z(-\lambda)) + \mathcal{A}_n(\lambda^2) + \mathcal{D}_n(\lambda^2) \tag{94}$$

which we shall use to find $t(\lambda)$. In terms of (89) and (90) this reads

$$t(\lambda)t(-\lambda) = (-1)^\rho\left(\delta_+^n + \delta_-^n\right) + x_+^n + x_-^n, \tag{95}$$

where $\delta_\pm = (b \pm a\varkappa\lambda)(d \mp c\varkappa\lambda)$; $\delta_+\delta_- = \delta(\lambda^2) = x_+x_-$. Introducing q taking the n values $\pi(2s+1-\rho)/n$, $s = 0, \ldots, n-1$, we can write (94) as

$$t(\lambda)t(-\lambda) = (-1)^n \prod_{\mathsf{q}} (e^{\mathrm{i}\mathsf{q}}\delta_+ - \tau(\lambda^2) + \mathrm{e}^{-\mathrm{i}\mathsf{q}}\delta_-) = (-1)^n \prod_{\mathsf{q}} (A(\mathsf{q})\lambda^2 - C(\mathsf{q}) + 2\mathrm{i}B(\mathsf{q})\lambda)$$

with

$$A(\mathsf{q}) = a^2c^2 - 2\varkappa ac\cos\mathsf{q} + \varkappa^2; \qquad B(\mathsf{q}) = (ad-bc)\sin\mathsf{q}; \\ C(\mathsf{q}) = 1 - 2(bd/\varkappa)\cos\mathsf{q} + b^2d^2/\varkappa^2. \tag{96}$$

Factorizing the polynomial in λ we obtain

$$t(\lambda)t(-\lambda) = (-1)^n \prod_{\mathsf{q}} A(\mathsf{q})(\lambda - s_{\mathsf{q}})(\lambda + s_{-\mathsf{q}}) \tag{97}$$

with

$$s_{\mathsf{q}} = \frac{1}{A(\mathsf{q})}(\sqrt{D(\mathsf{q})} - \mathrm{i}B(\mathsf{q})), \qquad D(\mathsf{q}) = A(\mathsf{q})C(\mathsf{q}) - B(\mathsf{q})^2, \tag{98}$$

(fixing the sign of $\sqrt{D(q)}$ requires a special convention, see [25]) and after some arguments we find the spectrum

$$t(\lambda) = (a^nc^n + (-1)^\rho\varkappa^n)\prod_{\mathsf{q}}(\lambda \pm s_{\mathsf{q}}), \tag{99}$$

where the signs are not yet fixed. Comparing the λ-independent term in (51)

$$t(\lambda) = 1 + (-1)^\rho b^nd^n/\varkappa^n + E_1\lambda + \cdots + E_{n-1}\lambda^{n-1} + \lambda^n(a^nc^n + (-1)^\rho\varkappa^n). \tag{100}$$

with the corresponding term in (99) shows that the number of minus signs in (99) must be even (odd) for the NS-sector $\rho = 0$ (R-sector $\rho = 1$).

It is useful to introduce the following notion: the eigenvalue (99) with all + signs is called to possess 'no quasi-particle' excitations. Each factor labeled by q having a minus sign is said to contribute the 'excitation of the q-quasi-momentum'. We shall accordingly label the minus signs by a set of variables $\sigma_{\mathsf{q}} \in \mathbb{Z}_2$, where for unexcited (excited) levels q we put $\sigma_{\mathsf{q}} = 0$ ($\sigma_{\mathsf{q}} = 1$). So instead of (99), we shall write more precisely

$$t^{(\rho)}(\lambda) = (a^nc^n + (-1)^\rho\varkappa^n)\prod_{\mathsf{q}}(\lambda + (-1)^{\sigma_{\mathsf{q}}}s_{\mathsf{q}}). \tag{101}$$

The corresponding eigenvectors have been considered for the inhomogeneous case in subsection 5.2.

6.2. Functional relation for the diagonal-to-diagonal Ising model transfer matrix

In this subsection we specialize the results of the previous subsection to the case of the diagonal-to-diagonal transfer matrix of the Ising model on a square lattice (15). So, we set $a = c, b = -d, \varkappa = 1$ and $\lambda = b/a$. Let us calculate the ingredients of the functional relation (94). We have $F_m(\lambda) = -(b - a\lambda)^2$. Therefore due to (60), $z(\lambda) = (-1)^n(b + a\lambda)^{2n}$, $z(b/a) = (-1)^n(2b)^{2n}$, $z(-b/a) = 0$ and the averaged L-operator (27) at $\lambda^2 = b^2/a^2$ becomes

$$\mathcal{L}_k(b^2/a^2) = \begin{pmatrix} 1 - b^2/a^2, & -b^2/a^2(a^2-b^2) \\ a^2 - b^2, & b^2(b^2-a^2) \end{pmatrix} = \begin{pmatrix} 1 \\ a^2 \end{pmatrix} \cdot (1 - b^2/a^2) \cdot (1, \;\; -b^2).$$

Hence

$$\mathcal{A}_n(b^2/a^2) + \mathcal{D}_n(b^2/a^2) = \operatorname{tr}\mathcal{T}_n(b^2/a^2) = (1 - b^2/a^2)^n(1 - a^2b^2)^n.$$

Substituting these expressions into (94), we get the following functional relation

$$t(b/a)t(-b/a) = (-1)^{\rho+n}(2b)^{2n} + (1 - b^2/a^2)^n(1 - a^2b^2)^n.$$

We want to compare this with the functional relation equation (7.5.5) in [43]:

$$V(K, L)V(L + \mathrm{i}\pi/2, -K)C = (2\mathrm{i}\sinh 2L)^n I + (-2\mathrm{i}\sinh 2K)^n R,$$

where C is the operator of translation, R is the operator of spin flip $\mathbf{V}_n$ and $V(K, L)$ is the transfer-matrix (13) with $K_x = 0, K_y = L, K_d = K, \mathrm{e}^{-2L} = b/a, \tanh K = ab$. Therefore, $V(K, L) = \exp(nL)\cosh^n K\,\mathbf{t}_n(b/a)$. Similar analysis gives $V(L+\mathrm{i}\pi/2, -K)C = \mathrm{i}^n\exp(nL)\cosh^n K\,\mathbf{t}_n(-b/a)$. Now taking into account that the eigenvalues of R are $(-1)^\rho$ and

$$2\sinh 2K = \frac{4ab}{1 - a^2b^2}, \qquad 2\sinh 2L = \frac{a^2 - b^2}{ab}, \qquad \frac{\exp(-2L)}{\cosh^2 K} = (1 - a^2b^2)b/a,$$

we see that both functional relations are identical.

6.3. *Ising model: Spectra and zeros of the $B_n(\lambda)$- and $t_n(\lambda)$-eigenvalue polynomials*

We now specialize further to the Ising case (14) as already advertised in subsection 2.2:

$$a_j = c_j = a, \qquad b_j = d_j = b, \qquad \varkappa_j = 1; \quad \forall j. \tag{102}$$

In the Ising case (102) the 2^n eigenvalues of (101) with (98) can be written (2^{n-1} in each sector $\rho = 0, 1$) as

$$t^{(\rho)}(\lambda) = (a^{2n} + (-1)^\rho)\prod_{\mathsf{q}}(\lambda + (-1)^{\sigma_{\mathsf{q}}}s_{\mathsf{q}}), \qquad s_{\mathsf{q}} = s_{-\mathsf{q}} = \sqrt{\frac{b^4 - 2b^2\cos\mathsf{q} + 1}{a^4 - 2a^2\cos\mathsf{q} + 1}}, \tag{103}$$

where the quasi-momentum q in each sector takes n values:

$$\mathsf{q} = \frac{2\pi}{n}m, \quad m \text{ integer for } \rho = 1 (R); \quad m \text{ half-integer for } \rho = 0 \text{ (NS)}. \tag{104}$$

Recall that we found from (100) that in the NS (R) sector, the eigenstates of $\mathbf{t}(\lambda)$ have an even (odd) number of excitations: $\prod_{\mathsf{q}}(-1)^{\sigma_{\mathsf{q}}} = (-1)^\rho$.

For $\mathsf{q} = 0$ (this occurs for R-sector only) and $\mathsf{q} = \pi$ we define

$$s_0 = \frac{b^2 - 1}{a^2 - 1}, \qquad s_\pi = \frac{b^2 + 1}{a^2 + 1}. \tag{105}$$

$\mathsf{q} = \pi$ is in the R sector for n even. However, for n odd it is in the NS sector. The different presence of factors $(\lambda \pm s_0)$ and $(\lambda \pm s_\pi)$ in (103) for n even or odd often makes it necessary to consider the cases n-even and n-odd separately. In the following we shall reserve the notation λ_{q} for $\lambda_{\mathsf{q}} = (-1)^{\sigma_{\mathsf{q}}}s_{\mathsf{q}}$ and otherwise use s_{q} as defined in (103).

The zeros $\lambda_{n,k}$ of the $B_n(\lambda)$ eigenvalue polynomial are determined by (93), (89), (90):

$$\tau\left(\lambda_{n,k}^2\right) = 4\cos^2 q_{n,k}F(\lambda_{n,k})F(-\lambda_{n,k}), \qquad q_{n,k} = \pi k/n, \qquad k = 1, \ldots, n-1. \tag{106}$$

Since now

$$F(\lambda) = F(-\lambda) = b^2 - a^2\lambda^2; \qquad \tau(\lambda^2) = 1 + b^4 - (1 + a^4)\lambda^2, \tag{107}$$

we obtain

$$r_{n,k} = \sqrt{(b^4 - 2b^2\cos q_{n,k} + 1)/(a^4 - 2a^2\cos q_{n,k} + 1)} = s_{q_{n,k}}. \tag{108}$$

Observe that s_{q} and $r_{n,k}$ may coincide.

6.4. Ising model state vectors from Baxter equations

In order to obtain the eigenvectors of $\mathbf{t}(\lambda)$, we have to solve Baxter's equations. For our restricted parameters (102) we have $F(\lambda) = F(-\lambda)$ and the left and right Baxter equations (57) and (56) become identical. Omitting the superscripts L and R on Q_k and recalling $\lambda_{n,k} = -(-1)^{\rho_{n,k}} r_{n,k}$, $\rho_{n,k} = 0, 1$ we obtain:

$$t_n(\lambda_{n,k})Q_k(\rho_{n,k}) = \left(\frac{(-1)^{\rho}F^{n-1}(\lambda_{n,k})}{(\lambda_{n,k})^{n-1}\chi_k} + (-\lambda_{n,k})^{n-1}\chi_k F(\lambda_{n,k})\right) Q_k(\rho_{n,k}+1). \tag{109}$$

From (109) we get the following compatibility condition:

$$t(-r_{n,k})t(r_{n,k}) = (-1)^{n-1}\left(\frac{(-1)^{\rho}F^{n-1}(r_{n,k})}{(r_{n,k})^{n-1}\chi_k} + (-r_{n,k})^{n-1}\chi_k F(r_{n,k})\right)^2,$$

if $t(\lambda)$ is an eigenvalue from the sector ρ. If $(-1)^k = (-1)^{\rho+1}$ then the quasi-momentum $\mathsf{q} = q_{n,k}$ belongs to the sector ρ and for $r_{n,k} = s_{q_{n,k}}$ we have $t(-r_{n,k})t(r_{n,k}) = 0$. This implies a relation not depending on a particular $t(\lambda)$ and its ρ:

$$\chi_k^2 r_{n,k}^{2(n-1)} = (-1)^{n+k+1}F^{n-2}(r_{n,k}). \tag{110}$$

Although the eigenvalue polynomial $t(\lambda)$ is known from (103), to solve (109) for the $Q_k(\rho_{n,k})$ can meet a difficulty if $t_n(\lambda_{n,k})$ vanishes or if, due to (110), the big bracket on the right of (109) vanishes. All this can happen and we have to distinguish four cases (we suppress n and write just $r_k = r_{n,k}$ and $\rho_k = \rho_{n,k}$):

(i) $(-1)^{\rho} = (-1)^k$. This is the easy case, since from (104) and (106) $t^{\rho}(r_k) \neq 0$ and $t^{\rho}(-r_k) \neq 0$, and we may normalize and solve

$$Q_k^{\mathrm{L,R}}(0) = 1, \qquad Q_k^{\mathrm{L,R}}(1) = \frac{(-1)^{n-1}t^{\rho}(-r_k)}{2\chi_k r_k^{n-1}F(r_k)}.$$

The other three cases occur for $(-1)^{\rho} = (-1)^{k-1}$.

(ii) $t^{\rho}(r_k) \neq 0$, $t^{\rho}(-r_k) = 0$. $t^{\rho}(\lambda)$ contains a factor $(\lambda + r_k)^2$ (both $\mathsf{q} = \pm q_k$ not excited), we may normalize

$$Q_k^{\mathrm{L,R}}(0) = 1, \qquad Q_k^{\mathrm{L,R}}(1) = 0.$$

(iii) $t^{\rho}(r_k) = 0$, $t^{\rho}(-r_k) \neq 0$. $t^{\rho}(\lambda)$ contains a factor $(\lambda - r_k)^2$ (both $\mathsf{q} = \pm q_k$ are excited), we cannot choose $Q_k^{\mathrm{L,R}}(0) = 1$, but we may normalize

$$Q_k^{\mathrm{L,R}}(0) = 0, \qquad Q_k^{\mathrm{L,R}}(1) = 1.$$

(iv) $t^{\rho}(r_k) = t^{\rho}(-r_k) = 0$. $t^{\rho}(\lambda)$ contains $\left(\lambda^2 - r_k^2\right)$ (either $\mathsf{q} = +q_k$ or $\mathsf{q} = -q_k$ is excited): A L'Hôpital procedure, using a slight perturbation of (102) as described in [26], is required (to obtain eigenvectors of translation operator), leading to

$$Q_k^{\mathrm{R}}(0) = Q_k^{\mathrm{L}}(0) = 1, \qquad Q_k^{\mathrm{R}}(1) = -Q_k^{\mathrm{L}}(1) = \frac{(-1)^{n+\sigma_{q_k}+1}2\mathrm{i}\sin q_k t^{\rho}_{\breve{q}_k}(-r_k)}{n\chi_k r_k^{n-1}A(q_k)}$$

(observe that from the L'Hôpital-limit $Q_k^{\mathrm{R}}(1) = -Q_k^{\mathrm{L}}(1)$), where

$$t^{\rho}(\lambda) = t^{\rho}_{\breve{q}_k}(\lambda)\left(\lambda + (-1)^{\sigma_{q_k}}s_{q_k}\right)\left(\lambda - (-1)^{\sigma_{q_k}}s_{-q_k}\right), \qquad A(\mathsf{q}) = a^4 - 2a^2\cos\mathsf{q} + 1. \tag{111}$$

In the following we shall consider only the three cases which allow the normalization $Q_k^{\mathrm{L,R}}(0) = 1$. Case (iii) can be treated too, but requires a special treatment, which here we shall not enter. According to which case the corresponding eigenvalue polynomial belongs, let us define the sets $\breve{\mathcal{D}}^{(\rho)}$, $\widehat{\mathcal{D}}^{(\rho)}$, $\mathcal{D}^{(\rho)}$:

$k \in \breve{\mathcal{D}}^{(\rho)}$ if t^{ρ} has a factor $(\lambda + r_k)^2$, i.e. we have case (ii),

$k \in \widehat{\mathcal{D}}^{(\rho)}$ if t^{ρ} has a factor $(\lambda - r_k)^2$, case (iii), and

$k \in \mathcal{D}^{(\rho)}$ if t^{ρ} has a factor $(\lambda^2 - r_k^2)$, i.e. we have case (iv).

By $D = |\mathcal{D}|$ we denote the number of elements in $\mathcal{D} = \mathcal{D}^{(0)} \cup \mathcal{D}^{(1)}$, similarly for $\breve{\mathcal{D}}$, etc.

7. Calculation of the matrix elements of σ_n^z in the homogeneous Ising model

7.1. Explicit evaluation of the factors $\mathcal{N}(\rho')R_0(\rho')R(\rho')$ in (85)

We now start to evaluate (85) with (83) and (86) for the homogeneous Ising model where the parameters simplify drastically. Now

$$\zeta = b/a, \qquad r_0^2 = (a^2 - b^2)(a^{4n} - 1)/(a^4 - 1) \tag{112}$$

and $\mathbf{u}_n$ is represented by the Pauli σ_z.

We had agreed to consider initial states from the R-sector. Then for matrix elements of σ_z the final state must be NS. We specify the initial state by the momenta which are excited, i.e. by the σ_k which are one, analogously the final state. Excluding for the time being case (iii), we take $\widehat{\mathcal{D}}^{(\rho)}$ to be empty.

On the right of (83) we have to evaluate the factors $\mathcal{N}(\boldsymbol{\rho}')R_0(\boldsymbol{\rho}')R(\boldsymbol{\rho}')$. Let us start with $R_0(\boldsymbol{\rho}') = \prod_{l=1}^{n-1} Q_l^{(0)}(\rho_l)Q_l^{(1)}(\rho_l)$.

For any choice of excitations, always one of the factors $Q_l^{(0)}(\rho_l)$ or $Q_l^{(0)}(\rho_l)$ is from case (i) of subsection 6.4. Since we exclude for the moment case (iii), the other factors then must be from (ii) or (iv). So always $Q_l^{(0)}(0)Q_l^{(1)}(0) = 1$. For $l \in \breve{\mathcal{D}}$, case (ii), we have $Q_l^{(0)}(1)Q_l^{(1)}(1) = 0$ since either $Q_l^{(0)}(1) = 0$ or $Q_l^{(1)}(1) = 0$ depending on the parity of l. So, in (83) the summation reduces to the summation over ρ_l for $l \in \mathcal{D}$ only, with fixed $\rho_l = 0$ for $l \in \breve{\mathcal{D}}$.

$R_0(\boldsymbol{\rho}')$ receives non-trivial contributions from $Q_k(1)$ of cases (i) and (iv). However, these can be written in a simple way if we use the explicit formulae for $t^{(\rho)}(-r_k)$. For both values $\rho_l = 0, 1$ the result is

$$Q_l^{(0)}(\rho_l)Q_l^{(1)}(\rho_l) = (-1)^{(n-1)\rho_l}\frac{(-1)^{\rho_l}r_l + \xi_l}{r_l + \xi_l} \cdot \prod_{k\in\breve{\mathcal{D}}}\frac{(-1)^{\rho_l}r_l + r_k}{r_l + r_k}, \tag{113}$$

where we get different results according to whether s_0 or s_π or both (105) are excited:

$$\xi_l = \begin{cases} (-1)^{\sigma_0}\dfrac{b^2 - \mathrm{e}^{\mathrm{i}q}}{a^2 - \mathrm{e}^{\mathrm{i}q}} \\ (-1)^{\sigma_0}\dfrac{b^2\,\mathrm{e}^{\mathrm{i}q} - 1}{a^2 - \mathrm{e}^{\mathrm{i}q}} \end{cases} \text{for} \quad (-1)^{\sigma_0} = \pm(-1)^{\sigma_\pi}; \quad \tilde{q}_l = (-1)^{\sigma_{q_l}+|\mathcal{D}|+l}q_l. \tag{114}$$

Now, multiplying by $\mathcal{N}(\boldsymbol{\rho}')$, it is easy to see that the products $k \in \breve{\mathcal{D}}$ in (113) cancel (recall that $\rho_k = 0$ for $k \in \breve{\mathcal{D}}$) and we get finally

$$\mathcal{N}(\boldsymbol{\rho}) \cdot R_0(\boldsymbol{\rho}') = \prod_{l\in\mathcal{D}}(-1)^{\rho_l}\frac{(-1)^{\rho_l}r_l + \xi_l}{r_l + \xi_l}\prod_{m\in\mathcal{D},m>l}\frac{r_l + r_m}{(-1)^{\rho_l}r_l + (-1)^{\rho_m}r_m}. \tag{115}$$

In the calculation of $R(\rho')$ in (86) we have to insert our explicit expressions for $t^{(0)}(-\zeta_n)$ and $t^{(1)}(\zeta_n)$ from (103). Here, as already mentioned after (105), the cases of even n and odd n

give different formulae. For example, the factor $(\lambda - (-1)^{\sigma_\pi} s_\pi)$ is present only for R n even and NS n odd. So

$$\text{NS}, n \text{ odd:}\quad t^{(0)}(-\zeta) = (a^{2n}+1)(-\zeta + (-1)^{\sigma_\pi} s_\pi) \prod_{k\in\check{\mathcal{D}}^{(0)}} (-\zeta + r_k)^2 \prod_{l\in\mathcal{D}^{(0)}} \left(\zeta^2 - r_l^2\right), \tag{116}$$

(for even n omit the bracket with s_π), since in the NS-sector only odd k appear, and these fall into one of the classes (ii) and (iv), class (iii) being momentarily excluded. Analogously,

$$R, n \text{ odd:}\quad t^{(1)}(\zeta) = (a^{2n}-1)(\zeta + (-1)^{\sigma_0} s_0) \prod_{k\in\check{\mathcal{D}}^{(1)}} (\zeta + r_k)^2 \prod_{l\in\mathcal{D}^{(1)}} \left(\zeta^2 - r_l^2\right). \tag{117}$$

By slight manipulation we can move the ρ_l-dependent terms such that they appear only in one place each in the numerator and obtain

$$R^{(n\,\text{odd})}(\boldsymbol{\rho}') = \mathsf{R}\cdot\left\{(-1)^{\sigma_\pi}(a^2+1)\,(-\zeta+(-1)^{\sigma_\pi}s_\pi)\prod_{l\in\mathcal{D}}((-1)^{\rho_l}r_l+\zeta) - (-1)^{\sigma_0}\right.$$
$$\left.\times\,(a^2-1)\,(\zeta+(-1)^{\sigma_0}s_0)\prod_{l\in\mathcal{D}}((-1)^{\rho_l}r_l-\zeta)\right\}\cdot\prod_{k\in\check{\mathcal{D}}}((-1)^k\zeta+r_k) \tag{118}$$

with

$$\mathsf{R} = (\boldsymbol{\alpha\beta})^{-(n-1)/2}\,a^{n-1}(a^{4n}-1)/(a^4-1), \qquad \boldsymbol{\alpha} = a^2-b^2, \qquad \boldsymbol{\beta} = 1-a^2b^2. \tag{119}$$

The first term in the curly bracket comes from the NS-sector final state, the second from the R initial state. The formula for $R^{(n\,\text{even})}$ is similar.

7.2. *Summation, square of the matrix element*

Combining (115) with (118) the spin matrix element is given by a multiple sum over the components of $\boldsymbol{\rho}'$:

$$\frac{\langle\Phi_0|\sigma_n^z|\Phi_1\rangle}{\langle\tilde{\Psi}_{\mathbf{0}}|\tilde{\Psi}_{\mathbf{0}}\rangle} = \sum_{\boldsymbol{\rho}'\in\mathbb{Z}_2^{n-1}}\left(\mathcal{R}_+^{\nu}\prod_{l\in\mathcal{D}}((-1)^{\rho_l}r_l+\zeta) + \mathcal{R}_-^{\nu}\prod_{l\in\mathcal{D}}((-1)^{\rho_l}r_l-\zeta)\right)$$
$$\times\prod_{l\in\mathcal{D}}(-1)^{\rho_l}\frac{(-1)^{\rho_l}r_l+\xi_l}{r_l+\xi_l}\prod_{m\in\mathcal{D},m>l}\frac{r_l+r_m}{(-1)^{\rho_l}r_l+(-1)^{\rho_m}r_m}, \tag{120}$$

with some $\boldsymbol{\rho}'$-independent factors $\mathcal{R}_\pm^{\nu}$. The superscript ν is there to remind us that we have different expressions for n even and n odd, respectively. Now, in [27] it is shown that this sum can be performed, resulting in a factorized expression. As an example here we quote the summation formula for the multiple summation over ρ_l with $l\in\mathcal{D}$ if the dimension of $\mathcal{D}$ is odd and ξ_l defined by the upper formula of (114):

$$\sum_{\rho_l,l\in\mathcal{D}}\frac{\prod_{l\in\mathcal{D}}(-1)^{\rho_l}((-1)^{\rho_l}r_l+\xi_l)((-1)^{\rho_l}r_l+\zeta)}{\prod_{l<m,l,m\in\mathcal{D}}((-1)^{\rho_l}r_l+(-1)^{\rho_m}r_m)} = \mathcal{C}(b\pm a)$$
$$\times\left(\prod_{j\in\mathcal{D}}e^{\mathrm{i}\tilde{q}_j}\mp ab\right)\frac{\prod_{l\in\mathcal{D}}(2r_l/a)(a^2e^{\mathrm{i}\tilde{q}_l}-1)^{(D-1)/2}(e^{\mathrm{i}\tilde{q}_l}-a^2)^{(D-3)/2}}{\prod_{l,m\in\mathcal{D},l<m}(\pm(e^{\mathrm{i}\tilde{q}_l+\mathrm{i}\tilde{q}_m}-1))}, \tag{121}$$
$$\mathcal{C} = \boldsymbol{\alpha}^{-(D-1)(D-3)/4}\boldsymbol{\beta}^{-(D-1)^2/4}.$$

The case of $\mathcal{D}$ even is similar (see (52) of [27]).

In the following, we shall be interested in the product of the matrix elements of the spin operator between arbitrary periodic states, which does not depend on normalization of the left and right eigenstates, i.e. we want to calculate

$$\frac{\langle\Phi_0|\mathbf{u}_n|\Phi_1\rangle\langle\Phi_1|\mathbf{u}_n|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle\langle\Phi_1|\Phi_1\rangle}. \tag{122}$$

Taking the absolute squares, several factors in (121) can be rewritten, e.g.

$$|a^2\mathrm{e}^{\mathrm{i}\tilde{q}_l}-1|^2 = |\mathrm{e}^{\mathrm{i}\tilde{q}_l}-a^2|^2 = A(\tilde{q}_l) = a^4-2a^2\cos\tilde{q}_l+1, \tag{123}$$

$$|\mathrm{e}^{\mathrm{i}\tilde{q}_l+\mathrm{i}\tilde{q}_m}-1|^2 = \frac{r_m^2-r_l^2}{\boldsymbol{\alpha\beta}}A(\tilde{q}_m)A(\tilde{q}_l)\frac{\sin\frac{1}{2}(\tilde{q}_l+\tilde{q}_m)}{\sin\frac{1}{2}(\tilde{q}_l-\tilde{q}_m)} \tag{124}$$

and all factors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $A(\tilde{q}_m)$ cancel. So we get for arbitrary n and $\sigma_0=\sigma_\pi$:

$$\frac{\langle\Phi_0|\sigma_n^z|\Phi_1\rangle\langle\Phi_1|\sigma_n^z|\Phi_0\rangle}{\langle\tilde{\Psi}_{0,\mathbf{0}}|\tilde{\Psi}_{0,\mathbf{0}}\rangle^2} = \left(\lambda_\pi^2-\lambda_0^2\right)^{(D-\delta)/2}(\lambda_0+\lambda_\pi)^\delta\prod_{l\in\mathcal{D}}\frac{2r_l}{(\lambda_0+r_l)(\lambda_\pi+r_l)}$$
$$\times\prod_{l<m,l,m\in\mathcal{D}}\frac{r_l+r_m}{r_l-r_m}\cdot\frac{\sin\frac{1}{2}(\tilde{q}_l-\tilde{q}_m)}{\sin\frac{1}{2}(\tilde{q}_l+\tilde{q}_m)} \tag{125}$$

where $\delta=1$. In a similar way we can find the product of matrix elements in the case of $\sigma_0\neq\sigma_\pi$. The final result is (125) with $\delta=0$. Observe that the explicit appearance of excitations of type (ii), i.e. $k\in\breve{\mathcal{D}}$ has disappeared from our formula (recall that we still exclude $k\in\widehat{\mathcal{D}}$).

7.3. *Normalization of the periodic states, final result in terms of* λ_0, λ_π, r_k *and* $\tilde{q}_l$

In order to compare (125) to the results obtained by A Bugrij and O Lisovyy [17, 18] we change the normalization and calculate the ratio (122). To do this we have to divide (125) by

$$\langle\Phi_0|\Phi_0\rangle\langle\Phi_1|\Phi_1\rangle/\langle\tilde{\Psi}_{0,\mathbf{0}}|\tilde{\Psi}_{0,\mathbf{0}}\rangle^2. \tag{126}$$

However, formula (76) cannot be used directly in our degenerate Ising case (14). As in case (iv) we have first to go off the Ising point and consider $ad-bc=\eta$ and apply l'Hopital's rule for $\eta\to 0$. For n odd the result is

$$\frac{\langle\Phi_0|\Phi_0\rangle\langle\Phi_1|\Phi_1\rangle}{\langle\tilde{\Psi}_{0,\mathbf{0}}|\tilde{\Psi}_{0,\mathbf{0}}\rangle^2} = 2^{|\mathcal{D}|}\prod_{k=1}^{n}(2r_{n,k})\cdot\frac{\prod_{k\text{-odd}}(\lambda_\pi\pm r_{n,k})}{\prod_{k\text{-even}}(\lambda_\pi+r_{n,k})}\cdot\frac{\prod_{k\text{-even}}(\lambda_0\pm r_{n,k})}{\prod_{k\text{-odd}}(\lambda_0+r_{n,k})}$$
$$\times\frac{\prod_{k<l,k,l\text{-odd}}((r_{n,k}+r_{n,l})(\pm r_{n,k}\pm r_{n,l}))\prod_{k<l,k,l\text{-even}}((r_{n,k}+r_{n,l})(\pm r_{n,k}\pm r_{n,l}))}{\prod_{k\text{-odd},l\text{-even}}((\pm r_{n,k}+r_{n,l})(r_{n,k}\pm r_{n,l}))},$$

and similar for n even, see [26].

Including also the hitherto excluded case (iii), our final formula for the matrix element is

$$\frac{\langle\Phi_0|\sigma_n^z|\Phi_1\rangle\langle\Phi_1|\sigma_n^z|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle\langle\Phi_1|\Phi_1\rangle} = \left(\lambda_\pi^2-\lambda_0^2\right)^{(D-\delta)/2}(\lambda_0+\lambda_\pi)^\delta\prod_{\substack{l<m\\l,m\in\mathcal{D}}}\left(\frac{r_l+r_m}{r_l-r_m}\cdot\frac{\sin\frac{1}{2}(\tilde{q}_l-\tilde{q}_m)}{\sin\frac{1}{2}(\tilde{q}_l+\tilde{q}_m)}\right)$$
$$\times\frac{\Lambda_n}{2^D\prod_{k\in\overline{\mathcal{D}}}(\dot{+}2r_k)}\cdot\frac{\prod_{k\,\text{odd},l\,\text{even}}((\dot{-}r_k\dot{+}r_l)(\dot{+}r_k\dot{-}r_l))}{\prod_{k<l,k,l\,\text{odd}}((\dot{+}r_k\dot{+}r_l)(\dot{-}r_k\dot{-}r_l))\prod_{k<l,k,l\,\text{even}}((\dot{+}r_k\dot{+}r_l)(\dot{-}r_k\dot{-}r_l))}, \tag{127}$$

where

$$\Lambda_n = \frac{\prod_{k\in\overline{\mathcal{D}}^{(0)}}(\lambda_0 \dot{+} r_k)}{\prod_{k\in\overline{\mathcal{D}}^{(1)}}(\lambda_0 \dot{+} r_k)\prod_{k\in\mathcal{D}^{(1)}}\left(\lambda_0^2 - r_k^2\right)} \cdot \frac{\prod_{k\in\overline{\mathcal{D}}^{(1)}}(\lambda_\pi \dot{+} r_k)}{\prod_{k\in\overline{\mathcal{D}}^{(0)}}(\lambda_\pi \dot{+} r_k)\prod_{k\in\mathcal{D}^{(0)}}\left(\lambda_\pi^2 - r_k^2\right)}, \qquad \text{for odd } n,$$

$$\Lambda_n = \frac{\prod_{k\in\overline{\mathcal{D}}^{(0)}}(\lambda_0 \dot{+} r_k)(\lambda_\pi \dot{+} r_k)}{(\lambda_0 + \lambda_\pi)\prod_{k\in\overline{\mathcal{D}}^{(1)}}(\lambda_0 \dot{+} r_k)(\lambda_\pi \dot{+} r_k)\prod_{k\in\mathcal{D}^{(1)}}\left(\lambda_0^2 - r_k^2\right)\left(\lambda_\pi^2 - r_k^2\right)}, \qquad \text{for even } n.$$

Here we used a superimposed dot: $\dot{\pm} r_m$ as the short notation for r_m if $m \in \check{\mathcal{D}}$, for $\pm r_m$ if $m \in \mathcal{D}$ and for $-r_m$ if $m \in \widehat{\mathcal{D}}$, respectively. For composite sets of momentum levels k we write $\overline{\mathcal{D}} = \check{\mathcal{D}} \cup \widehat{\mathcal{D}}$, $\overline{\mathcal{D}}^{(0)} = \check{\mathcal{D}}^{(0)} \cup \widehat{\mathcal{D}}^{(0)}$, $\overline{\mathcal{D}}^{(1)} = \check{\mathcal{D}}^{(1)} \cup \widehat{\mathcal{D}}^{(1)}$.

7.4. Final result in terms of momenta

Let $\{\mathsf{q}_1, \mathsf{q}_2, \ldots, \mathsf{q}_K\}$ and $\{\mathsf{p}_1, \mathsf{p}_2, \ldots, \mathsf{p}_L\}$ be the sets of the momenta of the excitations presenting the states $|\Phi_0\rangle$ from the NS-sector and $|\Phi_1\rangle$ from the R-sector, respectively. After some lengthy but straightforward transformations of (127) we obtain

$$\frac{\langle\Phi_0|\sigma_n^z|\Phi_1\rangle\langle\Phi_1|\sigma_n^z|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle\langle\Phi_1|\Phi_1\rangle} = J(s_\pi + s_0)\left(s_\pi^2 - s_0^2\right)^{(K+L-1)/2}$$

$$\times \prod_{k=1}^{K} \frac{P^{\mathrm{NS}}_{\mathsf{q}_k} \prod_{\mathsf{q}\neq|\mathsf{q}_k|}^{\frac{\mathrm{NS}}{2}} N_{\mathsf{q},\mathsf{q}_k}}{\prod_{\mathsf{p}}^{\frac{\mathrm{R}}{2}} N_{\mathsf{p},\mathsf{q}_k}} \cdot \prod_{l=1}^{L} \frac{P^{\mathrm{R}}_{\mathsf{p}_l} \prod_{\mathsf{p}\neq|\mathsf{p}_l|}^{\frac{\mathrm{R}}{2}} N_{\mathsf{p},\mathsf{p}_l}}{\prod_{\mathsf{q}}^{\frac{\mathrm{NS}}{2}} N_{\mathsf{q},\mathsf{p}_l}} \cdot \frac{\prod_{k=1}^{K}\prod_{l=1}^{L} M_{\mathsf{q}_k,\mathsf{p}_l}}{\prod_{k<k'}^{K} M_{\mathsf{q}_k,\mathsf{q}_{k'}} \prod_{l<l'}^{L} M_{\mathsf{p}_l,\mathsf{p}_{l'}}}, \qquad (128)$$

where NS/2 (R/2) is the subset of quasi-momenta from NS (R) taking values in the segment $0 < \mathsf{q} < \pi$, NS/2 (R/2) containing q_k with odd k (even k):

$$M_{\alpha,\beta} = \frac{s_\alpha + s_\beta}{s_\alpha - s_\beta} \cdot \frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}, \qquad M_{\alpha,-\alpha} = \frac{s_\alpha^2\left(s_0^2 - s_\pi^2\right)}{\left(s_\pi^2 - s_\alpha^2\right)\left(s_0^2 - s_\alpha^2\right)},$$

$$N_{\alpha,\beta} = \frac{s_\alpha + s_\beta}{s_\alpha - s_\beta}, \qquad \mathcal{J} = \frac{\prod_{\mathsf{q}}^{\frac{\mathrm{NS}}{2}}(s_0 + s_{\mathsf{q}})}{\prod_{\mathsf{p}}^{\frac{\mathrm{R}}{2}}(s_0 + s_{\mathsf{p}})} \cdot \frac{\prod_{\mathsf{q}}^{\frac{\mathrm{NS}}{2}}\prod_{\mathsf{p}}^{\frac{\mathrm{R}}{2}}(s_{\mathsf{q}} + s_{\mathsf{p}})^2}{\prod_{\mathsf{q},\mathsf{q}'}^{\frac{\mathrm{NS}}{2}}(s_{\mathsf{q}} + s_{\mathsf{q}'})\prod_{\mathsf{p},\mathsf{p}'}^{\frac{\mathrm{R}}{2}}(s_{\mathsf{p}} + s_{\mathsf{p}'})}.$$

For n odd,

$$P^{\mathrm{NS}}_{\mathsf{q}} = \frac{s_{\mathsf{q}}}{(s_\pi - s_{\mathsf{q}})(s_0 + s_{\mathsf{q}})}, \qquad \mathsf{q} \neq \pi, \qquad P^{\mathrm{R}}_{\mathsf{p}} = \frac{s_{\mathsf{p}}}{(s_\pi + s_{\mathsf{p}})(s_0 - s_{\mathsf{p}})}, \qquad \mathsf{p} \neq 0,$$

$$P^{\mathrm{R}}_0 = P^{\mathrm{NS}}_\pi = \frac{1}{s_\pi + s_0}, \qquad J = \frac{\prod_{\mathsf{p}}^{\frac{\mathrm{R}}{2}}(s_\pi + s_{\mathsf{p}})}{\prod_{\mathsf{q}}^{\frac{\mathrm{NS}}{2}}(s_\pi + s_{\mathsf{q}})}\mathcal{J},$$

for n even,

$$P^{\mathrm{NS}}_{\mathsf{q}} = \frac{s_{\mathsf{q}}}{(s_\pi + s_{\mathsf{q}})(s_0 + s_{\mathsf{q}})}, \qquad P^{\mathrm{R}}_{\mathsf{p}} = \frac{s_{\mathsf{p}}}{(s_\pi - s_{\mathsf{p}})(s_0 - s_{\mathsf{p}})}, \qquad \mathsf{p} \neq 0, \pi,$$

$$P^{\mathrm{R}}_0 = -P^{\mathrm{NS}}_\pi = \frac{1}{s_\pi - s_0}, \qquad J = \frac{\prod_{\mathsf{p}}^{\frac{\mathrm{NS}}{2}}(s_\pi + s_{\mathsf{q}})}{\prod_{\mathsf{q}}^{\frac{\mathrm{R}}{2}}(s_\pi + s_{\mathsf{p}})}\mathcal{J}.$$

7.5. Bugrij–Lisovyy formula for the matrix elements

In [18] the following formula for the square of the matrix element of spin operator for the finite-size Ising model was conjectured:

$$\left|{}_{\rm NS}\langle \mathsf{q}_1,\mathsf{q}_2,\ldots,\mathsf{q}_K\left|\sigma_n^z\right|\mathsf{p}_1,\mathsf{p}_2,\ldots,\mathsf{p}_L\rangle_{\rm R}\right|^2$$

$$= \xi\xi_T \prod_{k=1}^{K} \frac{\prod_{\mathsf{q}\neq\mathsf{q}_k}^{\rm NS}\sinh\frac{\gamma(\mathsf{q}_k)+\gamma(q)}{2}}{n\prod_{\mathsf{p}}^{\rm R}\sinh\frac{\gamma(\mathsf{q}_k)+\gamma(\mathsf{p})}{2}} \prod_{l=1}^{L} \frac{\prod_{\mathsf{p}\neq\mathsf{p}_l}^{\rm R}\sinh\frac{\gamma(\mathsf{p}_l)+\gamma(\mathsf{p})}{2}}{n\prod_{\mathsf{q}}^{\rm NS}\sinh\frac{\gamma(\mathsf{p}_l)+\gamma(\mathsf{q})}{2}} \cdot \left(\frac{t_y-t_y^{-1}}{t_x-t_x^{-1}}\right)^{(K-L)^2/2}$$

$$\times \prod_{k<k'}^{K} \frac{\sin^2\frac{\mathsf{q}_k-\mathsf{q}_{k'}}{2}}{\sinh^2\frac{\gamma(\mathsf{q}_k)+\gamma(\mathsf{q}_{k'})}{2}} \prod_{l<l'}^{L} \frac{\sin^2\frac{\mathsf{p}_l-\mathsf{p}_{l'}}{2}}{\sinh^2\frac{\gamma(\mathsf{p}_l)+\gamma(\mathsf{p}_{l'})}{2}} \prod_{\substack{1\leqslant k\leqslant K\\ 1\leqslant l\leqslant L}} \frac{\sinh^2\frac{\gamma(\mathsf{q}_k)+\gamma(\mathsf{p}_l)}{2}}{\sin^2\frac{\mathsf{q}_k-\mathsf{p}_l}{2}}. \tag{129}$$

In this formula the states are labeled by the momenta of the excitations. The factors in front of the right-hand side of (129) are defined by

$$\xi = ((\sinh 2K_x \sinh 2K_y)^{-2} - 1)^{1/4}, \qquad \xi_T = \left(\frac{\prod_{\mathsf{q}}^{\rm NS}\prod_{\mathsf{p}}^{\rm R}\sinh^2\frac{\gamma(\mathsf{q})+\gamma(\mathsf{p})}{2}}{\prod_{\mathsf{q},\mathsf{q}'}^{\rm NS}\sinh\frac{\gamma(\mathsf{q})+\gamma(\mathsf{q}')}{2}\prod_{\mathsf{p},\mathsf{p}'}^{\rm R}\sinh\frac{\gamma(\mathsf{p})+\gamma(\mathsf{p}')}{2}}\right)^{1/4},$$

where $\gamma(\mathsf{q})$ is the energy of the excitation with quasi-momentum q:

$$\cosh\gamma(\mathsf{q}) = \frac{\left(t_x+t_x^{-1}\right)\left(t_y+t_y^{-1}\right)}{2\left(t_x^{-1}-t_x\right)} - \frac{t_y-t_y^{-1}}{t_x-t_x^{-1}}\cos\mathsf{q}, \tag{130}$$

and $t_x = \tanh K_x$, $t_y = \tanh K_y$.

Formula (129) can easily be derived from (128) if one takes into account the identification of parameters (14). In particular we have $t_x = ab$, $t_y = (a-b)/(a+b)$ and the relation

$$e^{\gamma(\mathsf{q})} = \frac{as_{\mathsf{q}}+b}{as_{\mathsf{q}}-b} \tag{131}$$

between the energy $\gamma(\mathsf{q})$ of the excitation with quasi-momentum q and the corresponding zero s_{q} of the $t(\lambda)$-eigenvalue polynomial (103). The following formulae give the correspondence between the different parts of (129) and (128):

$$\frac{\xi\xi_T}{\sinh\frac{1}{2}(\gamma(0)+\gamma(\pi))}\left(\frac{t_y-t_y^{-1}}{t_x-t_x^{-1}}\right)^{1/2} = J, \qquad \frac{\sinh^2\frac{\gamma(\alpha)+\gamma(\beta)}{2}}{\sin^2\frac{\alpha-\beta}{2}} = -\frac{t_y-t_y^{-1}}{t_x-t_x^{-1}}M_{\alpha,\beta}.$$

$$\frac{\prod_{\mathsf{q}\neq\mathsf{q}_k}^{\rm NS}\sinh\frac{\gamma(\mathsf{q}_k)+\gamma(\mathsf{q})}{2}}{n\prod_{\mathsf{p}}^{\rm R}\sinh\frac{\gamma(\mathsf{q}_k)+\gamma(\mathsf{p})}{2}} = \frac{s_0+s_\pi}{\sinh\frac{\gamma(0)+\gamma(\pi)}{2}}\,\frac{P_{\mathsf{q}_k}^{\rm NS}\prod_{\mathsf{q}\neq|\mathsf{q}_k|}^{\frac{\rm NS}{2}}N_{\mathsf{q},\mathsf{q}_k}}{\prod_{\mathsf{p}}^{\frac{\rm R}{2}}N_{\mathsf{p},\mathsf{q}_k}},$$

$$\frac{\prod_{\mathsf{p}\neq\mathsf{p}_l}^{\rm R}\sinh\frac{\gamma(\mathsf{p}_l)+\gamma(\mathsf{p})}{2}}{n\prod_{\mathsf{q}}^{\rm NS}\sinh\frac{\gamma(\mathsf{p}_j)+\gamma(\mathsf{q})}{2}} = \frac{s_0+s_\pi}{\sinh\frac{\gamma(0)+\gamma(\pi)}{2}}\,\frac{P_{\mathsf{p}_l}^{\rm R}\prod_{\mathsf{p}\neq|\mathsf{p}_l|}^{\frac{\rm R}{2}}N_{\mathsf{p},\mathsf{p}_l}}{\prod_{\mathsf{q}}^{\frac{\rm NS}{2}}N_{\mathsf{q},\mathsf{p}_l}},$$

for more details, see [27].

8. Matrix elements for the diagonal-to-diagonal transfer matrix and for the quantum Ising chain in a transverse field

In this section we derive the matrix elements of the spin operator between eigenvectors of the diagonal-to-diagonal transfer matrix for the Ising model on a square lattice (see section 2.2). In this case the parameters are given by (15). As has been explained there, if we vary the parameters a and b in such a way to have fixed $(a^2-b^2)/(1-a^2b^2) = 1/\mathsf{k}'$, the eigenvectors (and therefore matrix elements) will not change. So we fix $a = c = \mathsf{k}'^{-1/2}$ and $b = d = 0$. Expanding the transfer matrix (3) with such parameters we obtain

$$\mathbf{t}_n(\lambda) = \mathbf{1} - \frac{2\lambda}{\mathsf{k}'}\widehat{\mathcal{H}} + \cdots, \qquad \widehat{\mathcal{H}} = -\frac{1}{2}\sum_{k=1}^{n}\left(\sigma_k^z\sigma_{k+1}^z + \mathsf{k}'\sigma_k^x\right),$$

where $\widehat{\mathcal{H}}$ is the Hamiltonian of the periodic quantum Ising chain in a transverse field. From (103) we get the spectrum of this Hamiltonian:

$$\mathcal{E} = -\frac{1}{2}\sum_{\mathsf{q}} \pm\varepsilon(\mathsf{q}), \tag{132}$$

where the energies of the quasi-particle excitations are

$$\varepsilon(\mathsf{q}) = (1 - 2\mathsf{k}'\cos\mathsf{q} + \mathsf{k}'^2)^{1/2} = \left((\mathsf{k}'-1)^2 + 4\mathsf{k}'\sin^2\frac{\mathsf{q}}{2}\right)^{1/2}, \qquad \mathsf{q} \neq 0, \pi,$$
$$\varepsilon(0) = \mathsf{k}' - 1, \qquad \varepsilon(\pi) = \mathsf{k}' + 1.$$

In (132), the sign $+/-$ in the front of $\varepsilon(\mathsf{q})$ corresponds to the absence/presence of the excitation with the momentum q. The NS-sector includes the states with an even number of excitations, the R-sector those with an odd number of excitations. The momentum q runs over the same set as in (103). Since we have $a = c$ and $b = d$, the formula (128) with $s_{\mathsf{q}} = \mathsf{k}'/\varepsilon(\mathsf{q})$ for matrix elements for σ_n^z can be applied. After some simplification we get the analog of (129), now for the quantum Ising chain:

$$|_{\mathrm{NS}}\langle \mathsf{q}_1, \mathsf{q}_2, \ldots, \mathsf{q}_K|\sigma_m^z|\mathsf{p}_1, \mathsf{p}_2, \ldots, \mathsf{p}_L\rangle_{\mathrm{R}}|^2 = \mathsf{k}'^{\frac{(K-L)^2}{2}}\xi\xi_T\prod_{k=1}^{K}\frac{\mathrm{e}^{\eta(\mathsf{q}_k)}}{n\varepsilon(\mathsf{q}_k)}\prod_{l=1}^{L}\frac{\mathrm{e}^{-\eta(\mathsf{p}_l)}}{n\varepsilon(\mathsf{p}_l)}$$
$$\times\prod_{k<k'}^{K}\left(\frac{2\sin\frac{\mathsf{q}_k-\mathsf{q}_{k'}}{2}}{\varepsilon(\mathsf{q}_k)+\varepsilon(\mathsf{q}_{k'})}\right)^2\prod_{l<l'}^{L}\left(\frac{2\sin\frac{\mathsf{p}_l-\mathsf{p}_{l'}}{2}}{\varepsilon(\mathsf{p}_l)+\varepsilon(\mathsf{p}_{l'})}\right)^2\prod_{k=1}^{K}\prod_{l=1}^{L}\left(\frac{\varepsilon(\mathsf{p}_l)+\varepsilon(\mathsf{q}_k)}{2\sin\frac{\mathsf{p}_l-\mathsf{q}_k}{2}}\right)^2, \tag{133}$$

where

$$\xi = (\mathsf{k}'^2 - 1)^{\frac{1}{4}}, \qquad \xi_T = \frac{\prod_{\mathsf{q}}^{\mathrm{NS}}\prod_{\mathsf{p}}^{\mathrm{R}}(\varepsilon(\mathsf{q})+\varepsilon(\mathsf{p}))^{\frac{1}{2}}}{\prod_{\mathsf{q},\mathsf{q}'}^{\mathrm{NS}}(\varepsilon(\mathsf{q})+\varepsilon(\mathsf{q}'))^{\frac{1}{4}}\prod_{\mathsf{p},\mathsf{p}'}^{\mathrm{R}}(\varepsilon(\mathsf{p})+\varepsilon(\mathsf{p}'))^{\frac{1}{4}}}$$

and

$$\mathrm{e}^{\eta(\mathsf{q})} = \frac{\prod_{\mathsf{q}'}^{\mathrm{NS}}(\varepsilon(\mathsf{q})+\varepsilon(\mathsf{q}'))}{\prod_{\mathsf{p}}^{\mathrm{R}}(\varepsilon(\mathsf{q})+\varepsilon(\mathsf{p}))}.$$

Formally, all these formulae are correct for the paramagnetic phase where $\mathsf{k}' > 1$, and for the ferromagnetic phase where $0 \leqslant \mathsf{k}' < 1$. But for the case $0 \leqslant \mathsf{k}' < 1$ it is natural to redefine the energy of zero-momentum excitation as $\varepsilon(0) = 1 - \mathsf{k}'$ to be positive. From (132), this change of the sign of $\varepsilon(0)$ in the ferromagnetic phase leads to a formal change between absence–presence of zero-momentum excitation in the labeling of eigenstates. Therefore the number of the excitations in each sector (NS and R) becomes even. Direct calculation shows that the change of the sign of $\varepsilon(0)$ in (133) can be absorbed to obtain formally the same formula (133), but with new $\varepsilon(0)$ and even L (the number of the excitations in the R-sector) and new $\xi = (1 - \mathsf{k}'^2)^{1/4}$.

Formulae (129) and (133) allow one to reobtain well-known formulae for the Ising model, e.g. the spontaneous magnetization [1, 2]. Indeed, for the quantum Ising chain in the ferromagnetic phase ($0 \leqslant \mathsf{k}' < 1$) and in the thermodynamic limit $n \to \infty$ (when the energies of $|\mathrm{vac}\rangle_{\mathrm{NS}}$ and $|\mathrm{vac}\rangle_{\mathrm{R}}$ coincide, giving the degeneration of the ground state), we have $\xi_T \to 1$ and therefore the spontaneous magnetization $_{\mathrm{NS}}\langle\mathrm{vac}|\sigma_m^z|\mathrm{vac}\rangle_{\mathrm{R}} = \xi^{1/2} = (1 - \mathsf{k}'^2)^{1/8}$.

9. Conclusions

We have shown that finite-size state vectors of the Ising (and generalized Ising) model can be obtained using the method of separation of variables and solving explicitly Baxter equations. The Ising model is treated as a special $N = 2$ case of the $\mathbb{Z}_N$-Baxter–Bazhanov–Stroganov $\tau^{(2)}$-model. Finite-size spin matrix elements between arbitrary states are calculated by sandwiching the operators between the explicit form of the state vectors. For the standard Ising case this gives a proof of the fully factorized formula for the form factors (129) conjectured previously by Bugrij and Lisovyy.

We also extend this result to obtain a factorized formula for the matrix elements of the finite-size Ising quantum chain in a transverse field. We show how specific local operators can be expressed in terms of global elements of the monodromy matrix. The truncated functional relation guaranteeing non-trivial solutions of the Baxter equation is compared to Baxter's [43] Ising model functional relation.

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